EXPLANATION OF INDEPENDENCE

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Preface

The original inspiration and motivation for this thesis came from [Sch96]—an inexperienced author's attempt to write down in a short period of time everything he knew and many things he did not understand, yet containing a few new ideas. With all due respect for the author of [Sch96], I always felt that I could do better than that. Over many years of work—some of it on mathematics, but much more on a paid job as a software developper—the original inspiration together with input from more recent research by Alf Onshuus and Itay Ben-Yaacov was gradually transformed into the present dissertation.

Contents

Introduction

Independence relations[1] generalise concepts such as linear independence in vector spaces or transcendence in fields to much more general complete theories. (In an even more general context than covered in this thesis, they also generalise orthogonality in Hilbert spaces and stochastic independence.)

More complicated notions such as orthogonality of types are built on top of independence. Also, important dividing lines between theories are often defined by means of additional axioms which an independence relation may or may not satisfy. Therefore the notion of independence relation is certainly among the most fundamental notions in stability theory. Independence relations satisfying certain additional properties were studied in [Mak84], [HH84] and [KP97]. An independence relation for o-minimal theories is more or less explicit in [PS86] and in [Pil86]. Recently all important independence relations were unified by work of Thomas Scanlon, Alf Onshuus and Clifton Ealy.

In this thesis I study independence relations in a systematic way. For the purposes of this introduction let us say that a relation $\mathop{\smile\hskip-0.9em^|\ }$ is a pre-independence relation if $\mathop{\smile\hskip-0.9em^|\ }$ satisfies the first five axioms for independence relations as well as strong finite character (cf. Definitions 1.1 and 2.1). I generalise Saharon Shelah's idea of passing from dividing to forking as follows: Whenever $\mathop{\smile\hskip-0.9em^|\ }$, a candidate for being an independence relation, is in fact a pre-independence relation, then $\mathop{\smile\hskip-0.9em^|\ }^{*}$, derived from $\mathop{\smile\hskip-0.9em^|\ }$ as in Definition 1.16, is a better candidate in the sense that $\mathop{\smile\hskip-0.9em^|\ } = \mathop{\smile\hskip-0.9em^|\ }^{*}$ if $\mathop{\smile\hskip-0.9em^|\ }$ is already an independence relation, and $\mathop{\smile\hskip-0.9em^|\ }^{*}$ is always a pre-independence relation satisfying the extension axiom. (Hence local character is the only axiom that need not hold for $\mathop{\smile\hskip-0.9em^|\ }^{*}$.)

I show that Shelah-dividing independence $\mathop{\smile\hskip-0.9em^|\ }^{\mathrm{d}}$ and M-dividing independence $\mathop{\smile\hskip-0.9em^|\ }^{\mathrm{M}}$ from [Sch96] are pre-independence relations. For certain sets Ω of pairs of formulas I also define Ω-dividing $\mathop{\smile\hskip-0.9em^|\ }^{\Omega}$ (Shelah-dividing localised to Ω, cf. Definition 2.11) and show that it is a pre-independence relation. Ξ and

[1] An independence relation is the same thing as a notion of independence as defined by Kim and Pillay.

Ξ_{M}, two such sets Ω, are defined in Definitions 2.9 and 2.35. I show that, if $\mathop{\smile\hskip-0.9em\vert}^{\mathrm{f}}$ denotes Shelah-forking independence and $\mathop{\smile\hskip-0.9em\vert}^{\text{þ}}$ denotes thorn-forking independence, we have $\mathop{\smile\hskip-0.9em\vert}^{\mathrm{f}} = \mathop{\smile\hskip-0.9em\vert}^{\mathrm{d}*} = \mathop{\smile\hskip-0.9em\vert}^{\Xi*}$ and $\mathop{\smile\hskip-0.9em\vert}^{\text{þ}} = \mathop{\smile\hskip-0.9em\vert}^{\mathrm{M}*} = \mathop{\smile\hskip-0.9em\vert}^{\Xi_{\mathrm{M}}*}$. Since $\mathop{\smile\hskip-0.9em\vert}^{\mathrm{M}}$ and $\mathop{\smile\hskip-0.9em\vert}^{\Xi_{\mathrm{M}}}$ have simpler definitions than thorn-dividing, this helps to understand thorn-forking.

In the following I go into more detail and also describe some other related results.

M-dividing, and a simple definition of thorn-forking

M-dividing from [Sch96] is defined as follows: $A \mathop{\smile\hskip-0.9em\vert}^{\mathrm{M}}_{C} B$ holds iff for all sets C' such that $C \subseteq C' \subseteq \operatorname{acl}(BC)$ we have $\operatorname{acl}(AC') \cap \operatorname{acl}(BC') = \operatorname{acl} C'$. Note that with this definition we need to take care whether we evaluate algebraic closure acl in T or in T^{eq}. (The same is true for strong dividing and thorn-dividing, but not for Shelah-dividing.) In [Sch96] it was observed that M-dividing is closely related to 'modular pairs' in the lattice of algebraically closed sets, and that $\mathop{\smile\hskip-0.9em\vert}^{\mathrm{M}}$ as evaluated on the real elements of a pregeometric theory (e. g., strongly minimal or o-minimal) is the familiar notion of independence. Here I continue the study of M-dividing by proving:

- $\mathop{\smile\hskip-0.9em\vert}^{\mathrm{M}}$ is an independence relation iff $\mathop{\smile\hskip-0.9em\vert}^{\mathrm{M}}$ is symmetric. (This answers a question implicit in [Sch96].)

- Thorn-forking is the notion of forking corresponding to M-dividing, i. e., $\mathop{\smile\hskip-0.9em\vert}^{\text{þ}} = \mathop{\smile\hskip-0.9em\vert}^{\mathrm{M}*}$.

- If there is an independence relation $\mathop{\smile\hskip-0.9em\vert}$ for T^{eq} satisfying the condition $a \mathop{\smile\hskip-0.9em\vert}_{C} a \implies a \in \operatorname{acl}^{\mathrm{eq}} C$, then T is rosy and $\mathop{\smile\hskip-0.9em\vert}^{\text{þ}}$ is the coarsest independence relation for T^{eq} satisfying this condition.

The last result is dual to the situation with Shelah-forking, which, in a simple theory, is the finest independence relation.

Thorn-forking via localised Shelah-dividing

Forking is traditionally defined via local dividing, i. e., dividing of formulas. The notion of k-dividing of a formula as it appears in Byunghan Kim's treatment of simple theories can be seen as a more thorough localisation. In the same sense, dividing of a formula φ with respect to a k-inconsistency witness ψ for φ (introduced by Itay Ben-Yaacov) is even more radically local.

Back-porting some of Ben-Yaacov's ideas into the elementary context, I examine generalised local dividing with respect to a set Ω of 'inconsistency

pairs' (φ, ψ) where ψ is an inconsistency witness for φ. The only interesting cases I know are $\Omega = \Xi$ (the set of *all* inconsistency pairs) and $\Omega = \Xi_{\mathrm{M}}$ as defined in Definition 2.35. In the second chapter I show:

- $\mathop{\smile\!\!\!\!\!\!\mid}^{\Xi} = \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}}$, hence $\mathop{\smile\!\!\!\!\!\!\mid}^{\Xi*} = \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{f}}$, so Shelah-forking is a special case of Ω-forking.
- $\mathop{\smile\!\!\!\!\!\!\mid}^{\Xi_{\mathrm{M}}*} = \mathop{\smile\!\!\!\!\!\!\mid}^{\flat}$, so thorn-forking is a special case of Ω-forking.

I also define local D_Δ-ranks of types for finite $\Delta \subseteq \Omega$, isolate two technical conditions which Ω may satisfy ('transitivity' and 'normality') and prove:

- Ξ and Ξ_{M} are transitive and normal.
- If Ω is transitive and normal, then $\mathop{\smile\!\!\!\!\!\!\mid}^{\Omega}$ is a pre-independence relation.
- If Ω is transitive and normal and all D_Δ-ranks are finite, then $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}^{\Omega*}_{C} B$ holds iff $\mathrm{D}_\Delta(\bar{a}/BC) = \mathrm{D}_\Delta(\bar{a}/C)$.

Kernels and canonical bases

The weak canonical base of a type over an algebraically closed set is the smallest algebraically closed subset over which the type is free. In the third chapter I show that this concept is closely related to thorn-forking:

- If an independence relation (satisfying the anti-reflexivity condition $a \mathop{\smile\!\!\!\!\!\!\mid}_{C} a \implies a \in \operatorname{acl} C$) has weak canonical bases, then it is thorn-forking independence.

As important tools for studying (weak) canonical bases I define the kernel and the algebraic kernel of a sequence of indiscernibles: The (algebraic) kernel of an infinite sequence of indiscernibles is the greatest subset of its definable (resp. algebraic) closure over which it is still indiscernible. I show:

- Every infinite sequence of indiscernibles has a kernel and an algebraic kernel, and they are invariant under 'collinearity.'
- For $\mathop{\smile\!\!\!\!\!\!\mid}^{\flat}$ to have weak canonical bases the following condition is sufficient: $A \mathop{\smile\!\!\!\!\!\!\mid}^{\flat}_{C_1} B$, $A \mathop{\smile\!\!\!\!\!\!\mid}^{\flat}_{C_2} B$ and $C_1, C_2 \subseteq B$ together imply $A \mathop{\smile\!\!\!\!\!\!\mid}^{\flat}_{\operatorname{acl} C_1 \cap \operatorname{acl} C_2} B$.
- If $\mathop{\smile\!\!\!\!\!\!\mid}^{\flat}$ has weak canonical bases, then the weak canonical base of a type can be computed as the algebraic kernel of an arbitrary Morley sequence.

- In a stable theory the canonical base of a stationary type can be computed as the kernel of its Morley sequence. In simple theories the situation is slightly more complicated.
- If a sequence of indiscernibles has a canonical base in the sense of Buechler, then the canonical base coincides with the kernel.

Some old results

The author of [Sch96] never formally published his results. Some of them are generalised in Chapter 3 or treated in exercises. I feel that two of them should be mentioned here:

- Let T be a simple theory and let T' be a reduct of T that has elimination of hyperimaginaries. If $C = \mathrm{acl}^{\mathrm{eq}} C$ in T and $A \mathop{\smile\!\!\!\!\!\!|}_C B$ holds in T, then $A \mathop{\smile\!\!\!\!\!\!|}_C B$ holds in T' (Exercise 3.5).
- Let T be a simple theory with elimination of hyperimaginaries. T is 1-based iff the lattice of algebraically closed sets is modular (Exercise 3.29).

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Preliminaries

Readers are assumed to be acquainted with the culture of stability theory. It should be no surprise to them that we work in a big saturated model of a complete consistent theory. That we compute definable closure dcl and algebraic closure acl either by means of definable or algebraic formulas, or, equivalently, by means of automorphisms of the big model. That we work with many-sorted theories, such as T^{eq}, whenever we feel like it. And that we work with indiscernible sequences, which are always implicitly assumed to be infinite.

It should not be hard to learn these things from the first pages of a book like [Pil96].

Chapter 1

Abstract independence

In this chapter an axiomatic treatment of independence relations for a complete consistent first-order theory is presented. Some properties of forking and thorn-forking are proved in this context. Thorn-forking is defined in a new way—via M-dividing, a new notion introduced in Section 1.5.

While the geometric theory of forking is usually based on a combinatorial foundation, we will see that a geometric treatment is possible from the beginning. This approach is faster and probably more comprehensible than the usual combinatorial one, but we get slightly weaker results. We will improve them in the next chapter by means of the usual, more combinatorial, methods.

The exposition in this chapter is self-contained apart from the general cultural background of stability theory: Except in the notes at the end of each section, no other knowledge from stability theory is assumed. Tuples of elements ($\bar{a}, \bar{b}, \bar{c}, \dots$) or of variables ($\bar{x}, \bar{y}, \bar{z}, \dots$) are allowed to be infinite unless mentioned otherwise. When a formula is written $\varphi(\bar{x})$ it means that each of its free variables appears in the (possibly infinite) tuple $\bar{x}$. There is an endless supply of formal variables that we can use in types. $\mathrm{S}^*(B)$ denotes the class of complete types over B in arbitrarily long sequences of variables.

I write AB for $A \cup B$, and for any tuple $\bar{a} = (a_0, a_1, \dots,)$ I will abuse notation by writing $\bar{a}$ for the set $\{a_0, a_1, \dots,\}$ as well. $A \equiv_B A'$ means that there is an automorphism of the big models that fixes B pointwise and maps the set A to the set A'. $\bar{a} \equiv_B \bar{a}'$ means that there is an automorphism fixing B pointwise and mapping the tuple $\bar{a}$ to the tuple $\bar{a}'$. $(A, B) \equiv_C (A', B')$ means that there is an automorphism fixing C pointwise that maps A to A' and B to B'.

1.1 Just a bunch of silly axioms?

We will call a ternary relation $\mathop{\smile\!\!\!\!\!\mid}$ between (small) subsets of the big model an *independence relation* if it satisfies the axioms of the following definition:

Definition 1.1. *The following are the* axioms for independence relations:

(invariance)
If $A \mathop{\smile\!\!\!\!\!\mid}_C B$ and $(A', B', C') \equiv (A, B, C)$, then $A' \mathop{\smile\!\!\!\!\!\mid}_{C'} B'$.

(monotonicity)
If $A \mathop{\smile\!\!\!\!\!\mid}_C B$, $A' \subseteq A$ and $B' \subseteq B$, then $A' \mathop{\smile\!\!\!\!\!\mid}_C B'$.

(base monotonicity)
Suppose $D \subseteq C \subseteq B$. If $A \mathop{\smile\!\!\!\!\!\mid}_D B$, then $A \mathop{\smile\!\!\!\!\!\mid}_C B$.

(transitivity)
Suppose $D \subseteq C \subseteq B$. If $B \mathop{\smile\!\!\!\!\!\mid}_C A$ and $C \mathop{\smile\!\!\!\!\!\mid}_D A$, then $B \mathop{\smile\!\!\!\!\!\mid}_D A$.

(normality)
$A \mathop{\smile\!\!\!\!\!\mid}_C B$ implies $AC \mathop{\smile\!\!\!\!\!\mid}_C B$.

(extension)
If $A \mathop{\smile\!\!\!\!\!\mid}_C B$ and $\hat{B} \supseteq B$, then there is $A' \equiv_{BC} A$ such that $A' \mathop{\smile\!\!\!\!\!\mid}_C \hat{B}$.
(Equivalently, by invariance, there is $\hat{B}' \equiv_{BC} \hat{B}$ such that $A \mathop{\smile\!\!\!\!\!\mid}_C \hat{B}'$.)

(finite character)
If $A_0 \mathop{\smile\!\!\!\!\!\mid}_C B$ for all finite $A_0 \subseteq A$, then $A \mathop{\smile\!\!\!\!\!\mid}_C B$.

(local character)
For every A there is a cardinal $\kappa(A)$ with the following property:
For any set B there is a subset $C \subseteq B$ of cardinality $|C| < \kappa(A)$ such that $A \mathop{\smile\!\!\!\!\!\mid}_C B$.

Definition 1.2. *An independence relation is* strict *if it also satisfies the following axiom:*

(anti-reflexivity)
$a \mathop{\smile\!\!\!\!\!\mid}_B a$ implies $a \in \operatorname{acl} B$.

For a first example, we need look no further than the coarsest[1] relation of all: the trivial relation that holds for all triples. It satisfies all of the above

[1] Generalising common usage for equivalence relations and topologies, we say that a relation $\mathop{\smile\!\!\!\!\!\mid}$ is finer than $\mathop{\smile\!\!\!\!\!\mid}'$, and $\mathop{\smile\!\!\!\!\!\mid}'$ coarser than $\mathop{\smile\!\!\!\!\!\mid}$, if $A \mathop{\smile\!\!\!\!\!\mid}_C B$ implies $A \mathop{\smile\!\!\!\!\!\mid}'_C B$.

axioms except anti-reflexivity. So the trivial relation is always a (non-strict) independence relation.

In practice, when I say 'by transitivity' I will often mean the following stronger property (or a variant thereof):

Remark 1.3. *Let $\mathop{\smile\!\!\!\!\!\mid}$ be a relation satisfying monotonicity, transitivity and normality. Then $B \mathop{\smile\!\!\!\!\!\mid}_{CD} A$ and $C \mathop{\smile\!\!\!\!\!\mid}_D A$ together imply $BC \mathop{\smile\!\!\!\!\!\mid}_D A$.*

Proof. If $B \mathop{\smile\!\!\!\!\!\mid}_{CD} A$ and $C \mathop{\smile\!\!\!\!\!\mid}_D A$, then $BCD \mathop{\smile\!\!\!\!\!\mid}_{CD} A$ and $CD \mathop{\smile\!\!\!\!\!\mid}_D A$ by normality. Hence $BCD \mathop{\smile\!\!\!\!\!\mid}_D A$ by transitivity, so $BC \mathop{\smile\!\!\!\!\!\mid}_D A$ by monotonicity. $\square$

Example 1.4. (Everywhere infinite forest)
Let T be the theory, in a signature with one binary relation E, of a non-empty undirected tree that branches infinitely in every node. Then T is complete, and the models of T are precisely the non-empty forests that branch infinitely in every node. In this theory, $\operatorname{acl} A$ is the set of all nodes that lie on a path between two elements of A.

Consider the following relation:

$$A \mathop{\smile\!\!\!\!\!\mid}_C B \quad \iff \quad \textit{every path from } A \textit{ to } B \textit{ meets } \operatorname{acl} C.$$

It is easy to see that $A \mathop{\smile\!\!\!\!\!\mid}_C B$ implies $AC \mathop{\smile\!\!\!\!\!\mid}_C B$ and $\operatorname{acl} A \mathop{\smile\!\!\!\!\!\mid}_C B$. Using this, it is not hard to check that $\mathop{\smile\!\!\!\!\!\mid}$ is a strict independence relation. The details are left to the reader as an exercise (Exercise 1.8).

If we look for more general strict independence relations with Exercise 1.7 in mind, we will tend to find *coarse* strict independence relations such as thorn-forking. In the next section we will introduce Morley sequences. These will motivate us to try another approach that will lead us to *fine* strict independence relations such as Shelah-forking.

But first we observe that $A \mathop{\smile\!\!\!\!\!\mid}_C B \iff B \mathop{\smile\!\!\!\!\!\mid}_C A$ holds both for $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{a}}$ from Exercise 1.7 below and for $\mathop{\smile\!\!\!\!\!\mid}$ from Example 1.4. In the next section we will examine whether this is an accident.

Exercises

Solutions for all exercises can be found in the appendix in Section A.2.

Exercise 1.5. (relations between the axioms, existence and symmetry)
Consider the following additional properties which a relation $\mathop{\smile\!\!\!\!\!\mid}$ may satisfy:

(existence) For any A, B and C there is $A' \equiv_C A$ such that $A' \mathop{\smile\!\!\!\!\!\mid}_C B$.

(symmetry) $A \mathop{\smile\!\!\!\!\!\!|}_C B \iff B \mathop{\smile\!\!\!\!\!\!|}_C A$

(i) Any relation that satisfies invariance, extension and symmetry also satisfies normality.

(ii) Any relation that satisfies invariance, extension and local character also satisfies existence.

(iii) Any relation that satisfies invariance, monotonicity, transitivity, normality, existence and symmetry also satisfies extension.

Exercise 1.6. (local character)

(i) Suppose the relation $\mathop{\smile\!\!\!\!\!\!|}$ satisfies invariance and the existence condition from Exercise 1.5. Let $\kappa(A) = (|T| + |A|)^+$. For any sets A and B there is $C_1 \subseteq B$ such that $A \mathop{\smile\!\!\!\!\!\!|}_{C_1} B$ and also a set C_2 such that $|C_2| \leq \kappa(A)$ and $A \mathop{\smile\!\!\!\!\!\!|}_{C_2} B$.

(ii) Suppose $\mathop{\smile\!\!\!\!\!\!|}$ is an independence relation. Let $\mathcal{A}$ be a set of finite subsets of the big model such that for every finite subset A of the big model there is $A' \in \mathcal{A}$ such that $A \equiv A'$. Let $\kappa = \sup_{A \in \mathcal{A}} \kappa(A)$. Show that for any sets A and B there is $C \subseteq B$ such that $|C| < \kappa + |A|^+$ and $A \mathop{\smile\!\!\!\!\!\!|}_C B$.

Exercise 1.7. (modularity and distributivity)

Consider the following relation:

$$A \mathop{\smile\!\!\!\!\!\!|^{\mathrm{a}}}_C B \iff \operatorname{acl}(AC) \cap \operatorname{acl}(BC) = \operatorname{acl} C.$$

(i) The relation $\mathop{\smile\!\!\!\!\!\!|^{\mathrm{a}}}$ satisfies the existence condition from Exercise 1.5 (hard).

(ii) The relation $\mathop{\smile\!\!\!\!\!\!|^{\mathrm{a}}}$ satisfies all axioms for strict independence relations except base monotonicity.

(iii) The relation $\mathop{\smile\!\!\!\!\!\!|^{\mathrm{a}}}$ satisfies base monotonicity (and is a strict independence relation) iff the lattice of algebraically closed subsets of the big model is *modular,* i. e., whenever A, B, C are algebraically closed sets such that $B \supseteq C$, the equation $B \cap \operatorname{acl}(AC) = \operatorname{acl}((B \cap A)C)$ holds.

(iv) An independence relation is *perfectly trivial* if $A \mathop{\smile\!\!\!\!\!\!|}_C B$ implies $A \mathop{\smile\!\!\!\!\!\!|}_{C'} B$ for all $C' \supseteq C$. Suppose $\mathop{\smile\!\!\!\!\!\!|^{\mathrm{a}}}$ is an independence relation. Show that $\mathop{\smile\!\!\!\!\!\!|^{\mathrm{a}}}$ is a perfectly trivial independence relation if and only if the lattice of algebraically closed sets is *distributive,* i. e., $\operatorname{acl}((A \cap B)C) = \operatorname{acl}((A \cap C)(B \cap C))$ holds for all algebraically closed sets A, B and C.

Exercise 1.8.

For Example 1.4, check that $A \mathop{\smile\!\!\!\!\!\!|}_C B$ implies $\operatorname{acl} A \mathop{\smile\!\!\!\!\!\!|}_C B$ and that $\mathop{\smile\!\!\!\!\!\!|}$ is a strict independence relation.

Notes

The symbol '$\mathop{\smile\!\!\!\!\!\!|}$' was first used for independence by Michael Makkai in [Mak84], but the symbol $\perp$ was used in a similar context in lattice theory much earlier.[2]

[2] If we apply the definition of $A \perp B$ from [Neu60], a book based on John von Neumann's work on lattice theory in the 1930s, to the lattice of algebraically closed sets it means

A possible pronunciation of $\mathop{\smile\hspace{-0.9em}\perp}$ is ‘anchor’.

Independence is often expressed using another notation that is equivalent to the $\mathop{\smile\hspace{-0.9em}\perp}$ notation: For $C \subseteq B$ and complete types $p(\bar{x}) \in \mathrm{S}(C)$ and $q(\bar{x}) \in \mathrm{S}(B)$, write $p \sqsubseteq q$ if $p \subseteq q$ and $\bar{a} \mathop{\smile\hspace{-0.9em}\perp}_C B$, where $\bar{a}$ realises q. An axiomatic characterisation of (classical) forking independence in a stable theory in terms of such a relation was discovered by Victor Harnik and Leo A. Harrington [HH84]. The next big break-through in this direction was the core result of Byunghan Kim’s dissertation [Kim96] (see also [KP97]): an axiomatic characterisation of (Shelah-)forking independence in a simple theory.

The terms ‘independence relation’ and ‘notion of independence’ are not (yet) standardised. Some authors include certain axioms of varying strength (boundedness or the amalgamation property) that make sure that if there is an independence relation at all, then it is unique and coincides with classical forking independence (and the theory is stable or simple, respectively). Apart from that, the axiomatic systems are usually equivalent to the system of Definition 1.1.

Some axioms appear in a slightly unusual form here. The transitivity axiom from [Mak84], for instance, was dualised (A on the right-hand side) and separated into transitivity and normality. (The term ‘normality’ is new.) Many authors use another variant sometimes called ‘full transitivity’. The terminology around extension and existence is also far from unified. This probably dates back to [Mak84], where ‘existence’ combines existence and extension into one axiom. Local character was strengthened so it is more useful when we do not have finite character.

For Example 1.4 cf. the note to Example 2.40 below. Exercise 1.7 is from [Sch96]; the definition of $\mathop{\smile\hspace{-0.9em}\perp}^{\mathrm{a}}$ was inspired by [Low94]. Exercise 1.5 presents three easy relations that hold between the axioms of independence relations and existence and symmetry. Proving a fourth relation is the main object of the next section, while the fact that these are the only relations is the subject of Section 1.6.

1.2 Morley sequences and symmetry

We now prove that every independence relation is symmetric. But we need some preparations first.

Proposition 1.9. *Let $\mathop{\smile\hspace{-0.9em}\perp}$ be an independence relation.*
If $A \mathop{\smile\hspace{-0.9em}\perp}_D BC$ and $B \mathop{\smile\hspace{-0.9em}\perp}_D C$, then $AB \mathop{\smile\hspace{-0.9em}\perp}_D C$.
Actually, it is sufficient that $\mathop{\smile\hspace{-0.9em}\perp}$ satisfies the first five axioms.

Proof. $A \mathop{\smile\hspace{-0.9em}\perp}_D BC$ implies $A \mathop{\smile\hspace{-0.9em}\perp}_D BCD$ by extension and invariance, hence $A \mathop{\smile\hspace{-0.9em}\perp}_{BD} BCD$ by base monotonicity, hence $A \mathop{\smile\hspace{-0.9em}\perp}_{BD} C$ by monotonicity. Com-

$A \mathop{\smile\hspace{-0.9em}\perp}^{\mathrm{a}}_{\emptyset} B$. Note that the the original context was a very special type of lattices which were, in particular, modular. $(A, B) \perp$ in [Wil39], when translated in the same way, means $A \mathop{\smile\hspace{-0.9em}\perp}^{\mathrm{M}}_{\emptyset} B$, cf. Definition 1.26.

bining this with $B \mathop{\smile\!\!\!\!\!\!\!\;\mid}_D C$, we get $AB \mathop{\smile\!\!\!\!\!\!\!\;\mid}_D C$ by transitivity (i. e., by monotonicity, transitivity, normality and Remark 1.3). $\square$

Definition 1.10. *Let $\mathop{\smile\!\!\!\!\!\!\!\;\mid}$ be a ternary relation.*
A $\mathop{\smile\!\!\!\!\!\!\!\;\mid}$-Morley sequence in *a type $p(\bar{x}) \in \mathrm{S}^*(B)$ over a set $C \subseteq B$ is a sequence of B-indiscernibles $(\bar{a}_i)_{i<\omega}$ such that $(\bar{a}_i)_{i<n} \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C \bar{a}_n$ for every $n < \omega$, and every $\bar{a}_i$ realises $p(\bar{x})$.*
A $\mathop{\smile\!\!\!\!\!\!\!\;\mid}$-Morley sequence for *a complete type $p(\bar{x}) \in \mathrm{S}^*(B)$ is a $\mathop{\smile\!\!\!\!\!\!\!\;\mid}$-Morley sequence in $p(\bar{x})$ over B.*

Recall our convention that tuples may be infinite. In most cases just a convenience, this is crucial in this section and the next one. The following consequence of the Erdős-Rado theorem is proved in [BY03a, Lemma 1.2] in the more general context of compact abstract theories (cats):

Fact 1.11. ***(Extracting a sequence of indiscernibles)***
Let B be a set of parameters and κ a cardinal. Then for any sequence $(\bar{a}_i)_{i<\beth_{(2^{|T|+|B|+\kappa})^+}}$ consisting of sequences of length $|\bar{a}_i| = \kappa$ there is a B-indiscernible sequence $(\bar{a}'_j)_{j<\omega}$ with the following property:
For every $k < \omega$ there are $i_0 < i_1 < \ldots < i_k < \kappa$ such that $\bar{a}_{i_0}, \bar{a}_{i_1}, \ldots, \bar{a}_{i_k} \equiv_B \bar{a}'_0, \bar{a}'_1, \ldots, \bar{a}'_k$.

Proposition 1.12. *Suppose $\mathop{\smile\!\!\!\!\!\!\!\;\mid}$ is an independence relation and $\bar{a} \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C B$. Then there is a Morley sequence in $\mathrm{tp}(\bar{a}/BC)$ over C.*
Actually, it is sufficient that $\mathop{\smile\!\!\!\!\!\!\!\;\mid}$ satisfies the first five axioms and extension.

Proof. Let $\bar{a}_0 = \bar{a}$. We choose a very big cardinal κ and use extension and transfinite induction to construct a sequence $(\bar{a}_i)_{i<\kappa}$ satisfying $\bar{a}_\alpha \equiv_{BC} \bar{a}_0$ and $\bar{a}_\alpha \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C (\bar{a}_i)_{i<\alpha}$ for all $\alpha < \kappa$. If κ has been chosen sufficiently big, we can extract a sequence of BC-indiscernibles $(\bar{a}'_i)_{i<\omega}$ such that for every $n < \omega$ there are indices $\alpha_0, \ldots, \alpha_n$ such that $\bar{a}'_0 \ldots \bar{a}'_n \equiv_{BC} \bar{a}_{\alpha_0} \ldots \bar{a}_{\alpha_n}$. Note that $\bar{a}'_n \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C (\bar{a}'_i)_{i<n}$ by monotonicity and invariance.

By compactness we can 'invert' the sequence $(\bar{a}'_i)_{i<\omega}$, i. e., find a new sequence $(\bar{a}''_i)_{i<\omega}$ such that $\bar{a}''_0 \ldots \bar{a}''_n \equiv_{BC} \bar{a}'_n \ldots \bar{a}'_0$ holds for every $n < \omega$. In particular, the new sequence is also indiscernible over BC. This new sequence satisfies $\bar{a}''_0 \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C (\bar{a}''_i)_{0<i<n}$ for all $n < \omega$. Hence $(\bar{a}''_i)_{i<n} \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C \bar{a}_n$ for all $n < \omega$ by repeated use of Proposition 1.9.

Thus $(\bar{a}''_i)_{i<\omega}$ is a Morley sequence in $\mathrm{tp}(\bar{a}/BC)$ over C. $\square$

Proposition 1.13. *Suppose $\mathop{\smile\!\!\!\!\!\!\!\;\mid}$ is an independence relation and there is a Morley sequence in $\mathrm{tp}(\bar{a}/BC)$ over C. Then $B \mathop{\smile\!\!\!\!\!\!\!\;\mid}_C \bar{a}$.*
Actually, it is sufficient that $\mathop{\smile\!\!\!\!\!\!\!\;\mid}$ satisfies the first five axioms, finite character and local character.

Proof. Let $(\bar{a}_i)_{i<\omega}$ be a Morley sequence in $\mathrm{tp}(\bar{a}/BC)$ over C.

Let κ be a regular cardinal number that is greater than or equal to $\kappa(B)$ as in the local character axiom. By compactness we can extend the sequence $(\bar{a}_i)_{i<\omega}$ to obtain a BC-indiscernible sequence $(\bar{a}_i)_{i<\kappa}$. Using finite character, we see that $(\bar{a}_i)_{i<\alpha} \mathop{\smile\!\!\!\!\!\!|}_C \bar{a}_\alpha$ for each $\alpha < \kappa$.

By local character there is a set $D \subseteq C(\bar{a}_i)_{i<\kappa}$ of cardinality $|D| < \kappa$ such that $B \mathop{\smile\!\!\!\!\!\!|}_D C(\bar{a}_i)_{i<\kappa}$. By regularity of κ there is an index $\alpha < \kappa$ such that already $D \subseteq C(\bar{a}_i)_{i<\alpha}$. Thus, by base monotonicity and monotonicity, $B \mathop{\smile\!\!\!\!\!\!|}_{C(\bar{a}_i)_{i<\alpha}} \bar{a}_\alpha$.

Combining the results of the last two paragraphs using transitivity (actually, Remark 1.3 and monotonicity), we get $B \mathop{\smile\!\!\!\!\!\!|}_C \bar{a}_\alpha$. Since $\bar{a}_\alpha \equiv_{BC} \bar{a}$ this implies $B \mathop{\smile\!\!\!\!\!\!|}_C \bar{a}$ by invariance. □

For later use I have carefully noted which axioms were actually needed to prove Propositions 1.12 and 1.13. For our immediate use of them in this section it would not have been necessary:

Theorem 1.14. *Every independence relation* $\mathop{\smile\!\!\!\!\!\!|}$ *is symmetric:*
For any A, B and C, $A \mathop{\smile\!\!\!\!\!\!|}_C B$ iff $B \mathop{\smile\!\!\!\!\!\!|}_C A$.

Proof. If $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_C B$, then there is a Morley sequence in $\mathrm{tp}(\bar{a}/BC)$ over C by Proposition 1.12. Hence $B \mathop{\smile\!\!\!\!\!\!|}_C \bar{a}$ by Proposition 1.13. □

From now on we may use symmetry implicitly when working with an independence relation. For the rest of this chapter, however, we focus on *finding* independence relations.

Example 1.15. (A theory with no strict independence relation)
Consider the following two-sorted theory T_0: T_0 has two sorts P and E, the elements of which are called 'points' and 'equivalence relations', and a single ternary relation $\sim \subseteq P \times P \times E$ written as $p \sim_e q$. The axioms of T_0 say that $\sim_e$ is an equivalence relation on the points for every $e \in E$.

Clearly every substructure of a model of T_0 is again a model of T_0. The signature of T_0 is finite and relational. Moreover, the class of finite models of T_0 satisfies the joint embedding property and the amalgamation property. So by [Hod93, Theorem 7.4.1] T_0 has a Fraïssé limit T^* which is ω-categorical, has quantifier elimination, and whose finite submodels are precisely the finite models of T_0^*.

Since $\operatorname{acl} A = A$ for all sets, $A \mathop{\smile\!\!\!\!\!\!|}_C B \iff A \cap B \subseteq C$ defines a strict independence relation for T. But there is no strict independence relation for T^{eq}:

Suppose $\mathop{\smile\!\!\!\!\!\!|}$ is an independence relation for T^{eq}. Let $a_0 \in P$ be a single point, and let $(a_i)_{i<\omega}$ be a Morley sequence for $\mathrm{tp}(a_0/\emptyset)$. Let $e \in E$ be an

equivalence relation such that $a_i \sim_e a_j$ for any $i, j < \omega$. Then $(a_i)_{i<\omega}$ is indiscernible over e.

Note that $(a_{2i}a_{2i+1})_{i<\omega}$ is a Morley sequence for $\mathrm{tp}(a_0a_1/\emptyset)$ and is also indiscernible over e. By Proposition 1.13, $e \mathop{\smile\!\!\!\!\!\!\mid}_{\emptyset} a_0a_1$ holds, so by base monotonicity, $e \mathop{\smile\!\!\!\!\!\!\mid}_{a_0} a_1$. On the other hand, $a_0 \mathop{\smile\!\!\!\!\!\!\mid}_{\emptyset} a_1$ also holds. Applying transitivity we get $a_0e \mathop{\smile\!\!\!\!\!\!\mid}_{\emptyset} a_1$. Using symmetry and base monotonicity, we get $a_0 \mathop{\smile\!\!\!\!\!\!\mid}_{e} a_1$.

But the equivalence class c of a_0 and a_1 under $\sim_e$ is (an element of T^{eq} and) definable both over a_0e and over a_1e. So $c \mathop{\smile\!\!\!\!\!\!\mid}_{e} c$. Since c is clearly not algebraic over e this contradicts anti-reflexivity. So $\mathop{\smile\!\!\!\!\!\!\mid}$ is not a strict independence relation for T^{eq}.

Notes

While the terms and techniques used in this section are not new, the specific treatment of abstract independence, and Theorem 1.14 in particular, seems to be new. Once you have the right set of axioms, it is somewhat implicit in the work of Byunghan Kim. I first met the technique of Proposition 1.13 in [Kim96].

Theorem 1.14 is made possible by the fact that transitivity is postulated on the left-hand side in Definition 1.1. With the usual transitivity axiom (on the right-hand side, i.e.: $A \mathop{\smile\!\!\!\!\!\!\mid}_{C} B$ and $A \mathop{\smile\!\!\!\!\!\!\mid}_{D} C$ implies $A \mathop{\smile\!\!\!\!\!\!\mid}_{D} B$ for $D \subseteq C \subseteq B$), Theorem 1.14 would not hold. That is why symmetry is traditionally included as an axiom for independence relations. This point is usually obscured by authors who combine monotonicity, base monotonicity and right-hand side transitivity into an axiom called 'full transitivity'. I was set on the right track by [BY03a, Corollary 1.9].

Example 1.15 was suggested to me by Martin Ziegler.

1.3 A theorem on abstract forking

We now show that, in a certain sense, the extension axiom comes free.

Definition 1.16. *For any invariant relation $\mathop{\smile\!\!\!\!\!\!\mid}$ we define a new relation $\mathop{\smile\!\!\!\!\!\!\mid}^{*}$ as follows:*

$$A \mathop{\smile\!\!\!\!\!\!\mid}^{*}_{C} B \iff \big(\text{for all } \hat{B} \supseteq B \text{ there is } A' \equiv_{BC} A \text{ s.t. } A' \mathop{\smile\!\!\!\!\!\!\mid}_{C} \hat{B} \big).$$

Note that $A \mathop{\smile\!\!\!\!\!\!\mid}^{*}_{C} B$ implies $A \mathop{\smile\!\!\!\!\!\!\mid}_{C} B$. Also, $\mathop{\smile\!\!\!\!\!\!\mid} = \mathop{\smile\!\!\!\!\!\!\mid}^{*}$ iff $\mathop{\smile\!\!\!\!\!\!\mid}$ satisfies the extension axiom. In analogy to the classical situation one might call $\mathop{\smile\!\!\!\!\!\!\mid}^{*}$ the notion of forking derived from the abstract notion of dividing given by $\mathop{\smile\!\!\!\!\!\!\mid}$.

If $\mathop{\smile\!\!\!\!\!\mid}$ already satisfies some of the axioms for independence relations, then there cannot be much harm (possibly losing finite character and local character) in passing from $\downarrow$ to $\downarrow^*$, but we get extension free:

Lemma 1.17.
If $\downarrow$ is a relation satisfying invariance and montonicity, then $\downarrow^*$ satisfies invariance, monotonicity and extension. If, moreover, $\downarrow$ satisfies one of the following axioms and properties, then $\downarrow^*$ also satisfies it: base monotonicity, transitivity, normality, anti-reflexivity, existence.

Proof. Invariance of $\downarrow^*$ is obvious.

Monotonicity: Suppose $A \downarrow^*_C B$, $A_0 \subseteq A$ and $B_0 \subseteq B$. Then for all $\hat{B} \supseteq B$ there is $A' \equiv_{BC} A$ such that $A' \downarrow_C \hat{B}C$. Let $A'_0 \subseteq A'$ correspond to $A_0 \subseteq A$. Then clearly $A'_0 \equiv_{B_0 C} A_0$ and $A'_0 \downarrow_C \hat{B}$. Thus $A_0 \downarrow^*_C B_0$ holds.

Extension: Suppose $\bar{a} \downarrow^*_C B$, where $\bar{a}$ is a possibly infinite tuple, and let $\hat{B} \supseteq B$ be any superset of B.

We first claim that there is a type $p(\bar{x}) \in \mathrm{S}^*(\hat{B}C)$, extending $\mathrm{tp}(\bar{a}/BC)$, such that for all cardinals κ there is a κ-saturated model $M \supseteq \hat{B}C$ and $\bar{a}' \models p(\bar{x})$ such that $\bar{a}' \downarrow_C M$.

If not, then for each $p(\bar{x}) \in \mathrm{S}^*(\hat{B}C)$ extending $\mathrm{tp}(\bar{a}/BC)$ there is a cardinal $\kappa(p)$ such that for no $\kappa(p)$-saturated model $M \supseteq \hat{B}C$ is there a tuple $\bar{a}' \models p$ satisfying $\bar{a}' \downarrow_C M$. Let κ be the supremum of the cardinals $\kappa(p)$, and let $M \supseteq \hat{B}C$ be κ-saturated. Then there is no $\bar{a}' \equiv_{BC} \bar{a}$ such that $\bar{a}' \downarrow_C M$. So we have found a contradiction to the definition of $\downarrow^*$ and thereby proved our claim.

Now choose $\bar{a}' \models p(\bar{x})$, where $p(\bar{x})$ is as in the claim. Then clearly $\bar{a}' \equiv_{BC} \bar{a}$, and $\bar{a}' \downarrow^*_C \hat{B}$ by monotonicity of $\downarrow$.

Base monotonicity: Suppose $A \downarrow^*_C B$ and $C \subseteq C' \subseteq B$. So for any $\hat{B} \supseteq B$ there is $A' \equiv_{BC} A$ such that $A' \downarrow_C \hat{B}C$. Base monotonicity of $\downarrow$ yields $A' \downarrow_{C'} \hat{B}C$, so $A' \downarrow_{C'} \hat{B}$ by monotonicity of $\downarrow$. Thus $A \downarrow^*_{C'} B$.

Transitivity: Here we work with an alternative definition of $\downarrow^*$, which is equivalent to Definition 1.16 by invariance of $\downarrow$:

$$A \mathop{\downarrow^*}_C B \iff \big(\text{ for all } \hat{B} \supseteq B \text{ there is } \hat{B}' \mathop{\equiv}_{BC} \hat{B} \text{ s.t. } A \mathop{\downarrow}_C \hat{B}'\big).$$

So suppose $D \subseteq C \subseteq B$, $B \downarrow^*_C A$ and $C \downarrow^*_D A$ hold, and $\hat{A} \supseteq A$. We need to show that $B \downarrow_D \hat{A}^*$ for some $\hat{A}^* \equiv_{AD} \hat{A}$.

Let $\hat{A}' \equiv_{AD} \hat{A}$ be such that $C \downarrow_D \hat{A}'$, and let $\hat{A}^* \equiv_{AC} \hat{A}'$ be such that $B \downarrow_C \hat{A}^*$. Note that $\hat{A}^* \equiv_{AD} \hat{A}$ and $C \downarrow_D \hat{A}^*$. By transitivity of $\downarrow$ we get $B \downarrow_D \hat{A}^*$.

Normality: Suppose $A \mathop{\smile\!\!\!\!\!\mid}^*_C B$ and $\hat{B} \supseteq B$. Let $A' \equiv_{BC} A$ be such that $A' \mathop{\smile\!\!\!\!\!\mid}_C \hat{B}$. Then also $A'C \mathop{\smile\!\!\!\!\!\mid}_C \hat{B}$ by normality of $\mathop{\smile\!\!\!\!\!\mid}$.

Anti-reflexivity: Trivial, since $A \mathop{\smile\!\!\!\!\!\mid}^*_C B$ implies $A \mathop{\smile\!\!\!\!\!\mid}_C B$.

Existence: Suppose we are given A, B, C. Since $\mathop{\smile\!\!\!\!\!\mid}$ satisfies existence by assumption, we have $A \mathop{\smile\!\!\!\!\!\mid}^*_C \emptyset$. Since $\mathop{\smile\!\!\!\!\!\mid}^*$ satisfies extension there is $A' \equiv_C A$ such that $A' \mathop{\smile\!\!\!\!\!\mid}^*_C B$. □

Theorem 1.18. *Suppose $\mathop{\smile\!\!\!\!\!\mid}$ satisfies the first five axioms for independence relations and also finite character. Suppose $\mathop{\smile\!\!\!\!\!\mid}^*$, derived from $\mathop{\smile\!\!\!\!\!\mid}$ as in Definition 1.16, has local character. Then $\mathop{\smile\!\!\!\!\!\mid}^*$ is an independence relation.*

Proof. It follows from Lemma 1.17 that $\mathop{\smile\!\!\!\!\!\mid}^*$ satisfies the first five axioms and extension. As local character holds by assumption, we need only prove that $\mathop{\smile\!\!\!\!\!\mid}^*$ satisfies finite character. We will prove some other facts on our way.

First note that $\mathop{\smile\!\!\!\!\!\mid}^*$ satisfies the conditions of Proposition 1.12.

Then note that $A \mathop{\smile\!\!\!\!\!\mid}^*_C B$ implies $A \mathop{\smile\!\!\!\!\!\mid}_C B$. Hence $\mathop{\smile\!\!\!\!\!\mid}$ also has local character, and therefore $\mathop{\smile\!\!\!\!\!\mid}$ satisfies the conditions of Proposition 1.13.

Using the two propositions we can show that $A \mathop{\smile\!\!\!\!\!\mid}^*_C B$ implies $B \mathop{\smile\!\!\!\!\!\mid}_C A$: If $\bar{a} \mathop{\smile\!\!\!\!\!\mid}^*_C B$, there is a $\mathop{\smile\!\!\!\!\!\mid}^*$-Morley sequence in $\mathrm{tp}(\bar{a}/BC)$ over C. This sequence is also a $\mathop{\smile\!\!\!\!\!\mid}$-Morley sequence in $\mathrm{tp}(\bar{a}/BC)$ over C, hence $B \mathop{\smile\!\!\!\!\!\mid}_C \bar{a}$.

It follows that $A \mathop{\smile\!\!\!\!\!\mid}^*_C B$ implies $B \mathop{\smile\!\!\!\!\!\mid}^*_C A$: Suppose $A \mathop{\smile\!\!\!\!\!\mid}^*_C B$ and $\hat{A} \supseteq A$. Since $\hat{A} \mathop{\smile\!\!\!\!\!\mid}^*_{AC} AC$ by local character and base monotonicity, we can use extension to find $\hat{A}' \equiv_{AC} \hat{A}$ such that $\hat{A}' \mathop{\smile\!\!\!\!\!\mid}^*_{AC} ABC$, hence $\hat{A}' \mathop{\smile\!\!\!\!\!\mid}^*_{AC} B$ by monotonicity. Combining this with $A \mathop{\smile\!\!\!\!\!\mid}^*_C B$, we get $\hat{A}' \mathop{\smile\!\!\!\!\!\mid}^*_C B$ by transitivity. This implies $B \mathop{\smile\!\!\!\!\!\mid}_C \hat{A}$. Thus $B \mathop{\smile\!\!\!\!\!\mid}^*_C A$.

Now we can prove that $\mathop{\smile\!\!\!\!\!\mid}^*$ has finite character. Suppose $\bar{a} \mathop{\smile\!\!\!\!\!\mid}^*_C B$ holds for all finite $\bar{a} \in A$. We need to show that $A \mathop{\smile\!\!\!\!\!\mid}^*_C B$. So suppose $\hat{B} \supseteq B$. Since $A \mathop{\smile\!\!\!\!\!\mid}^*_{BC} BC$ by local character and base monotonicity, we can obtain $A' \equiv_{BC} A$ such that $A' \mathop{\smile\!\!\!\!\!\mid}^*_{BC} \hat{B}$ using existence and monotonicity. By invariance, there is also $\hat{B}' \equiv_{BC} \hat{B}$ such that $A \mathop{\smile\!\!\!\!\!\mid}^*_{BC} \hat{B}'$. It suffices to show that $A \mathop{\smile\!\!\!\!\!\mid}_C \hat{B}'$. For every finite $\bar{a} \in A$ we have $\bar{a} \mathop{\smile\!\!\!\!\!\mid}^*_C B$ by assumption, and $\bar{a} \mathop{\smile\!\!\!\!\!\mid}^*_{BC} \hat{B}'$ by $A \mathop{\smile\!\!\!\!\!\mid}^*_{BC} \hat{B}'$ and monotonicity. Since $\mathop{\smile\!\!\!\!\!\mid}^*$ is symmetric we can combine these results using transitivity on the right-hand side. Thus we get $\bar{a} \mathop{\smile\!\!\!\!\!\mid}^*_C \hat{B}'$ for all finite $\bar{a} \in A$. Hence $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_C \hat{B}'$ for all finite $\bar{a} \in A$. Since $\mathop{\smile\!\!\!\!\!\mid}$ has finite character, this implies $A \mathop{\smile\!\!\!\!\!\mid}_C \hat{B}'$. □

Notes

The traditional way to define independence in stability theory is by first defining a notion of 'dividing' and then deriving a notion of 'forking'. Both steps are usually

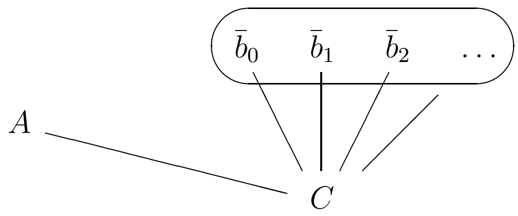

Figure 1.1: An attempt to illustrate the definition of $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}}$.

done with reference to individual formulas. While I have not seen Definition 1.16 in this form before (except for special cases in [Sch96]), it merely makes the step from dividing to forking explicit, while expressing it in semantic rather than syntactic terms. Thus it generalises the relation between classical dividing and classical forking and between thorn-dividing and thorn-forking.

Theorem 1.18 was probably not stated in this generality before. One reason is the fact that the local character axiom in its strong form (no finiteness condition on A) is needed to get $A \mathop{\smile\!\!\!\!\!\!\mid}^{*}_{B} B$ for all A, B from local character and base monotonicity. The proof of finite character in Theorem 1.18 is a bit contrived. Note that in Chapter 2 (Lemma 2.2) we will prove a shortcut for it in case $\mathop{\smile\!\!\!\!\!\!\mid}$ actually has *strong* finite character.

1.4 A theorem on Shelah-forking

Definition 1.19. *The relation* $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}}$ (Shelah-dividing independence) *is defined by*

$$A \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}}_{C} B \iff \left(\begin{array}{l} \textit{for any sequence of } C\textit{-indiscernibles } (\bar b_i)_{i<\omega} \textit{ s.t. } \bar b_0 \in BC\textit{:} \\ \exists\, A' \equiv_{BC} A \textit{ s.t. the sequence is } A'C\textit{-indiscernible} \end{array} \right).$$

The relation $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{f}}$ (Shelah-forking independence) *is defined by* $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{f}} = \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}*}$*, i.e.:*

$$A \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{f}}_{C} B \iff \left(\textit{for all } \hat B \supseteq B \textit{ there is } A' \underset{BC}{\equiv} A \textit{ s.t. } A' \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}}_{C} \hat B \right).$$

These definitions are equivalent to dividing and forking as they were originally defined by Saharon Shelah to study stable theories (cf. Exercise 1.24 below). The definition of $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}}$ can be seen as motivated by the following remark:

Remark 1.20.
If $\mathop{\smile\!\!\!\!\!\!\mid}$ *is any independence relation, then* $A \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}}_{C} B$ *implies* $A \mathop{\smile\!\!\!\!\!\!\mid}_{C} B$.

Proof. Suppose $A \mathop{\smile\hspace{-0.9em}\mid\hspace{0.4em}}^{\mathrm{d}}_{C} \bar{b}$. Let $(\bar{b}_i)_{i<\omega}$ be a $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.4em}}$-Morley sequence for $\mathrm{tp}(\bar{b}/C)$. This exists by Proposition 1.12, since $\bar{b} \mathop{\smile\hspace{-0.9em}\mid\hspace{0.4em}}_{C} C$. We may assume that $\bar{b}_0 = \bar{b}$. Since $A \mathop{\smile\hspace{-0.9em}\mid\hspace{0.4em}}^{\mathrm{d}}_{C} \bar{b}$ there is $A' \equiv_{\bar{b}C} A$ such that the sequence $(\bar{b}_i)_{i<\omega}$ is $A'C$-indiscernible. By Proposition 1.13, it now follows that $A \mathop{\smile\hspace{-0.9em}\mid\hspace{0.4em}}_{C} \bar{b}$. $\square$

Of course this remark implies that $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.4em}}^{\mathrm{d}} = \mathop{\smile\hspace{-0.9em}\mid\hspace{0.4em}}^{\mathrm{f}}$ whenever $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.4em}}^{\mathrm{f}}$ is an independence relation. But we still need $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.4em}}^{\mathrm{f}}$ for technical reasons.

Definition 1.21. *A complete consistent first-order theory T is* simple *if $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.4em}}^{\mathrm{f}}$ is an independence relation for T.*

We will see in Chapter 2 that T is simple iff T^{eq} is simple.

Lemma 1.22. *The relation $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.4em}}^{\mathrm{d}}$ of Shelah-dividing independence always satisfies the first five axioms for independence relations and finite character. It also satisfies anti-reflexivity.*

Proof. *Invariance* and *monotonicity* are obvious.

Base monotonicity: Suppose $A \mathop{\smile\hspace{-0.9em}\mid\hspace{0.4em}}^{\mathrm{d}}_{C} B$ and $C \subseteq C' \subseteq B$. Let $(\bar{b}_i)_{i<\omega}$ be a sequence of C'-indiscernibles with $\bar{b}_0 \in B = BC$. Let $\bar{c}'$ be an enumeration of C'. Then also $\bar{b}_0\bar{c}' \in BC$, and the sequence $(\bar{b}_i\bar{c}')_{i<\omega}$ is also C-indiscernible. Hence there is $A' \equiv_{\bar{b}_0\bar{c}'} A$ such that $(\bar{b}_i\bar{c}')_{i<\omega}$ is $A'C$-indiscernible. Thus $(\bar{b}_i)_{i<\omega}$ is $A'C'$-indiscernible.

Transitivity: Suppose $D \subseteq C \subseteq B$, $B \mathop{\smile\hspace{-0.9em}\mid\hspace{0.4em}}^{\mathrm{d}}_{C} A$ and $C \mathop{\smile\hspace{-0.9em}\mid\hspace{0.4em}}^{\mathrm{d}}_{D} A$. Let $(\bar{a}_i)_{i<\omega}$ be any sequence of D-indiscernibles with $\bar{a}_0 \in AD$.

By $C \mathop{\smile\hspace{-0.9em}\mid\hspace{0.4em}}^{\mathrm{d}}_{D} A$ there is $C' \equiv_{AD} C$ such that the sequence $(\bar{a}_i)_{i<\omega}$ is indiscernible over C'. Choose any set B' such that $(B', C') \equiv_{AD} (B, C)$. Then $B' \mathop{\smile\hspace{-0.9em}\mid\hspace{0.4em}}^{\mathrm{d}}_{C'} A$ holds by invariance. Hence there is $B'' \equiv_{AC'} B'$ such that the sequence is B''-indiscernible. And really, $B'' \equiv_{AD} B$.

Normality: Suppose $A \mathop{\smile\hspace{-0.9em}\mid\hspace{0.4em}}^{\mathrm{d}}_{C} B$. Let $(\bar{b}_i)_{i<\omega}$ be a sequence of C-indiscernibles such that $\bar{b}_0 \in BC$. By definition there is $A' \equiv_{BC} A$ such that the sequence is $A'C$-indiscernible. But then also $A'C \equiv_{BC} AC$.

Finite character: Let $\bar{a}$ be a possibly infinite tuple s. t. $\bar{a} \mathop{\not\smile\hspace{-0.9em}\mid\hspace{0.4em}}^{\mathrm{d}}_{C} B$. Let $p(\bar{x}) = \mathrm{tp}(\bar{a}/BC)$. Then there is a sequence $(\bar{b}_i)_{i<\omega}$ with $\bar{b}_0 \in BC$ such that the type extending $p(\bar{x})$ and the theory of the big model with constants for $BC(\bar{b}_i)_{i<\omega}$ and expressing that $(\bar{b}_i)_{i<\omega}$ is $\bar{a}C$-indiscernible is inconsistent. By compactness, a finite sub-tuple $\bar{a}_0$ of $\bar{a}$ is sufficient for this, so $\bar{a}_0 \mathop{\not\smile\hspace{-0.9em}\mid\hspace{0.4em}}^{\mathrm{d}}_{C} B$.

For *anti-reflexivity* suppose $a \notin \operatorname{acl} B$. Then there is a B-indiscernible sequence $(a_i)_{i<\omega}$ of distinct elements, with $a_0 = a$. This sequence witnesses that $a \mathop{\not\smile\hspace{-0.9em}\mid\hspace{0.4em}}^{\mathrm{d}}_{B} a$. $\square$

Theorem 1.23. *A theory T is simple if and only if $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.4em}}^{\mathrm{f}}$ has local character. If T is simple, $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.4em}}^{\mathrm{f}} = \mathop{\smile\hspace{-0.9em}\mid\hspace{0.4em}}^{\mathrm{d}}$, and this is the finest independence relation for T.*

Proof. For the equivalence, just apply Theorem 1.18 to $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}$. Now suppose T is simple. While $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_C B$ always implies $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_C B$, the converse is true by Remark 1.20. Hence $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}} = \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}$. Since $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ is an independence relation and $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}$ is finer than every independence relation by Remark 1.20, $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}} = \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}$ is the finest. □

Exercises

Exercise 1.24. (dividing and forking of formulas)

A formula $\varphi(\bar{x};\bar{b})$ *divides* over a set C if there is a finite number $k < \omega$ and a sequence $(\bar{b}_i)_{i<\omega}$ such that $\bar{b}_i \equiv_C \bar{b}$ holds for all $i < \omega$ and $\{\varphi(\bar{x};\bar{b}_i) \mid i < \omega\}$ is k-inconsistent. A formula *forks* over C if it implies a finite disjunction of formulas that divide over C.

(i) $\bar{a} \mathop{\smile\!\!\!\!\!\!/}^{\mathrm{d}}_C B$ iff there is a tuple $\bar{b} \in BC$ and a formula $\varphi(\bar{x};\bar{y})$ without parameters such that $\models \varphi(\bar{a};\bar{b})$ holds and $\varphi(\bar{x};\bar{b})$ divides over C.

(ii) $\bar{a} \mathop{\smile\!\!\!\!\!\!/}^{\mathrm{f}}_C B$ iff there is a tuple $\bar{b} \in BC$ and a formula $\varphi(\bar{x};\bar{y})$ without parameters such that $\models \varphi(\bar{a};\bar{b})$ holds and $\varphi(\bar{x};\bar{b})$ forks over C.

(iii) For simple T, a formula $\varphi(\bar{x};\bar{b})$ forks over a set C if and only if $\varphi(\bar{x};\bar{b})$ divides over C.

Exercise 1.25. (additional properties of $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}$)

(i) Every sequence of B-indiscernibles is also indiscernible over $\operatorname{acl} B$.

(ii) $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_C B$ implies $\operatorname{acl}(AC) \cap B \subseteq \operatorname{acl} C$.

(iii) $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_C B$ implies $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_C \operatorname{acl}(BC)$. So $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}$ always satisfies a weak variant of the extension axiom.

(iv) If $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_C B$ and $C \subseteq C' \subseteq \operatorname{acl}(BC)$, then $\operatorname{acl}(AC') \cap \operatorname{acl}(BC) = \operatorname{acl} C'$. Hence $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_C B$ implies $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}}_C B$, where $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}}$ is as defined in the next section.

Notes

Most of Lemma 1.22 can be found in [She90]. Transitivity of $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}$ in the general case is implicit in [Kim96], and most of Theorem 1.23 is also due to Byunghan Kim [Kim96]. I have not found the fact that Shelah-forking in a simple (or stable) theory is the finest independence relation stated outside [Sch96]. I think this is due to the fact that independence relations without any additional conditions are not usually an object of study.

1.5 A theorem on thorn-forking

Definition 1.26. *The relation* $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}}$ (M-dividing independence) *is defined by*

$$A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}}_C B \iff \left(\begin{array}{l} \text{for any } C' \text{ s.t. } C \subseteq C' \subseteq \operatorname{acl}(BC)\text{:} \\ \operatorname{acl}(AC') \cap \operatorname{acl}(BC') = \operatorname{acl} C' \end{array} \right).$$

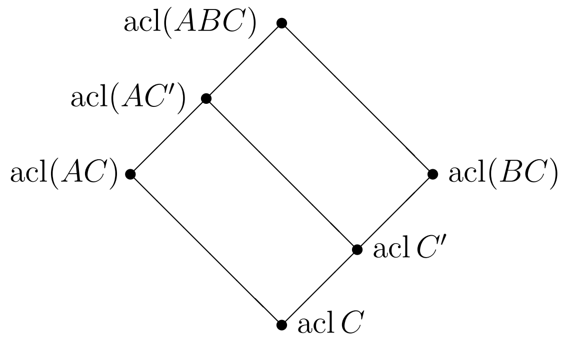

Figure 1.2: A lattice diagram illustrating the definition of $\mathop{\downarrow}^{\text{M}}$ in the lattice of algebraically closed sets. We have a map $\operatorname{acl} C' \mapsto \operatorname{acl}(AC')$ from the sublattice between $\operatorname{acl} C$ and $\operatorname{acl}(BC)$ to the sublattice between $\operatorname{acl}(AC)$ and $\operatorname{acl}(ABC)$. $A \mathop{\downarrow}^{\text{M}}_C B$ says that the map $D \mapsto D \cap \operatorname{acl}(BC)$ takes $D = \operatorname{acl}(AC')$ back to $\operatorname{acl} C'$.

The relation $\mathop{\downarrow}^{\text{þ}}$ (thorn-forking independence) *is defined by* $\mathop{\downarrow}^{\text{þ}} = \mathop{\downarrow}^{\text{M}*}$*, i.e.:*

$$A \mathop{\downarrow}^{\text{þ}}_{C} B \iff \Big(\textit{for all } \hat{B} \supseteq B \textit{ there is } A' \underset{BC}{\equiv} A \textit{ s.t. } A' \mathop{\downarrow}^{\text{M}}_{C} \hat{B} \Big).$$

It is shown in Section A.1 that this definition of $\mathop{\downarrow}^{\text{þ}}$ agrees on T^{eq} with thorn-forking independence as defined by Alf Onshuus. Here, however, we take the view that the definition of $\mathop{\downarrow}^{\text{þ}}$ is motivated by the following remark:

Remark 1.27.
If $\downarrow$ *is any strict independence relation, then* $A \mathop{\downarrow}_C B$ *implies* $A \mathop{\downarrow}^{\text{þ}}_C B$.

Proof. Suppose $\downarrow$ is a strict independence relation, $A \mathop{\downarrow}_C B$, and $\hat{B} \supseteq B$. We need to show that there is $A' \equiv_{BC} A$ such that $A' \mathop{\downarrow}^{\text{M}}_C \hat{B}$. So choose $A' \equiv_{BC} A$ such that $A' \mathop{\downarrow}_C \operatorname{acl}(\hat{B}C)$. For any D satisfying $C \subseteq D \subseteq \operatorname{acl}(\hat{B}C)$ we get $A' \mathop{\downarrow}_D \operatorname{acl}(\hat{B}C)$ by base monotonicity of $\downarrow$.

By extension and symmetry of $\downarrow$ there is a set $H \equiv_{A'D} \operatorname{acl}(A'D)$ that satisfies $H \mathop{\downarrow}_D \operatorname{acl}(\hat{B}C)$. Clearly $H = \operatorname{acl}(A'D)$, so $\operatorname{acl}(A'D) \mathop{\downarrow}_D \operatorname{acl}(\hat{B}C)$. Now by anti-reflexivity of $\downarrow$, $\operatorname{acl}(A'D) \cap \operatorname{acl}(\hat{B}CD) \subseteq \operatorname{acl} D$, so $\operatorname{acl}(A'D) \cap \operatorname{acl}(\hat{B}CD) = \operatorname{acl} D$. □

By comparing Remarks 1.20 and 1.27 one easily sees that $A \mathop{\downarrow}^{\text{f}}_C B$ implies $A \mathop{\downarrow}^{\text{þ}}_C B$, provided that a strict independence relation exists. Exercise 1.25 showed that even without this assumption a stronger statement is true: $A \mathop{\downarrow}^{\text{d}}_C B$ always implies $A \mathop{\downarrow}^{\text{M}}_C B$, hence $A \mathop{\downarrow}^{\text{f}}_C B$ always implies $A \mathop{\downarrow}^{\text{þ}}_C B$.

Definition 1.28. *A complete consistent first-order theory T is called* rosy *if* $\mathop{\smile\!\!\!\!\!\!\;|}^{\text{þ}}$ *is an independence relation for* T^{eq}.

Lemma 1.29. *The relation* $\mathop{\smile\!\!\!\!\!\!\;|}^{\text{M}}$ *of M-dividing independence always satisfies the first five axioms for independence relations and finite character. It also satisfies anti-reflexivity.*

Proof. *Invariance, monotonicity, normality* and *anti-reflexivity* are obvious.

Base monotonicity: Suppose $A \mathop{\smile\!\!\!\!\!\!\;|}^{\text{M}}_C B$ and $C \subseteq D \subseteq B$. Then for any D' satisfying $D \subseteq D' \subseteq \operatorname{acl}(BD)$ we also have $C \subseteq D' \subseteq \operatorname{acl}(BC)$. So $A \mathop{\smile\!\!\!\!\!\!\;|}^{\text{M}}_C B$ implies $\operatorname{acl}(AD') \cap \operatorname{acl}(BD') = \operatorname{acl} D'$. Hence $A \mathop{\smile\!\!\!\!\!\!\;|}^{\text{M}}_{D'} B$.

Transitivity: Suppose $D \subseteq C \subseteq B$, $B \mathop{\smile\!\!\!\!\!\!\;|}^{\text{M}}_{CD} A$ and $C \mathop{\smile\!\!\!\!\!\!\;|}^{\text{M}}_D A$. Then for any D' such that $D \subseteq D' \subseteq \operatorname{acl}(AD)$ we can compute:

$$\begin{aligned}
\operatorname{acl}(BD') \cap \operatorname{acl}(AD') &= \operatorname{acl}(BD') \cap \operatorname{acl}(ACD') \cap \operatorname{acl}(AD') \\
&= \operatorname{acl}(CD') \cap \operatorname{acl}(AD') && \big(\text{by } B \mathop{\smile\!\!\!\!\!\!\;|}^{\text{M}}_C A\big) \\
&= \operatorname{acl} D', && \big(\text{by } C \mathop{\smile\!\!\!\!\!\!\;|}^{\text{M}}_D A\big)
\end{aligned}$$

so $B \mathop{\smile\!\!\!\!\!\!\;|}^{\text{M}}_D A$ holds.

Finite character: Suppose $A \not\mathop{\smile\!\!\!\!\!\!\;|}^{\text{M}}_C B$. Let C' be such that $C \subseteq C' \subseteq \operatorname{acl}(BC)$ and $\operatorname{acl}(AC') \cap \operatorname{acl}(BC') \not\subseteq \operatorname{acl} C'$. Let $d \in (\operatorname{acl}(AC') \cap \operatorname{acl}(BC')) \setminus \operatorname{acl} C'$. Let $\bar{a} \in A$, finite, be such that $d \in \operatorname{acl}(\bar{a}C')$. Then clearly $\bar{a} \not\mathop{\smile\!\!\!\!\!\!\;|}^{\text{M}}_C B$. $\square$

Theorem 1.30. *The relation* $\mathop{\smile\!\!\!\!\!\!\;|}^{\text{þ}}$ *of thorn-forking independence is a strict independence relation if and only if it has local character, if and only if there is any strict independence relation at all. If* $\mathop{\smile\!\!\!\!\!\!\;|}^{\text{þ}}$ *is a strict independence relation, then it is the coarsest.*

Proof. To get the first equivalence, apply Theorem 1.18 to $\mathop{\smile\!\!\!\!\!\!\;|}^{\text{M}}$. If $\mathop{\smile\!\!\!\!\!\!\;|}$ is any strict independence relation, then, since $\mathop{\smile\!\!\!\!\!\!\;|}$ satisfies the local character axiom, so does $\mathop{\smile\!\!\!\!\!\!\;|}^{\text{þ}}$. If $\mathop{\smile\!\!\!\!\!\!\;|}^{\text{þ}}$ is a strict independence relation, then it is the coarsest by Remark 1.27. $\square$

In particular, if there is any strict independence relation for T^{eq}, then T is rosy. In a simple theory T, $\mathop{\smile\!\!\!\!\!\!\;|}^{\text{f}}$ is a strict independence relation for T^{eq} by Corollary 2.33 below, so every simple theory is rosy. Thus $\mathop{\smile\!\!\!\!\!\!\;|}^{\text{þ}}$ is the coarsest strict independence relation on T^{eq} while $\mathop{\smile\!\!\!\!\!\!\;|}^{\text{d}} = \mathop{\smile\!\!\!\!\!\!\;|}^{\text{f}}$ is the finest.

Note that the assumptions of Theorem 1.30 do not imply $\mathop{\smile\!\!\!\!\!\!\;|}^{\text{M}} = \mathop{\smile\!\!\!\!\!\!\;|}^{\text{þ}}$:

Example 1.31. (Everywhere infinite forest, continued from Example 1.4) It follows from Theorem 1.30 that $\mathop{\smile\!\!\!\!\!\!\;|}^{\text{þ}}$ is also a strict independence relation, and that $A \mathop{\smile\!\!\!\!\!\!\;|}_C B \implies A \mathop{\smile\!\!\!\!\!\!\;|}^{\text{þ}}_C B$. It is straightforward to check that the converse is also true, so $\mathop{\smile\!\!\!\!\!\!\;|} = \mathop{\smile\!\!\!\!\!\!\;|}^{\text{þ}}$. (Cf. Exercise 1.34.)

It is not hard to see that $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}}$ does not satisfy extension, so $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}} \neq \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}$: Let a and b be neighbours. Then $a \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}} b$. However, there is no $c \equiv_b a$ such that $a \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}} bc$: Either $c = a$, or b lies between a and c. In the first case, $a = c \in (\operatorname{acl} a \cap \operatorname{acl}(bc)) \setminus \operatorname{acl} \emptyset$, so $a \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}} bc$. In the second case, $b \in (\operatorname{acl}(ac) \cap \operatorname{acl}(bc)) \setminus \operatorname{acl} c$, so also $a \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}} bc$. Thus $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}}$ does not satisfy the extension axiom.

In some cases (most notably strongly minimal and o-minimal theories), þ-forking as defined on the real elements of T is an important tool for understanding the structure of models of T. In these cases þ-forking on T agrees with the restriction to T of þ-forking in T^{eq}. This is not the case in general, and the existence of a strict independence relation on the real elements of T *per se* does not imply any 'structure' that is more than superficial:

Example 1.32. (Thorn-forking must be computed in T^{eq} in general)
Let T be any complete consistent theory in a relational language. Consider the following theory T': The language of T' is the language of T together with a new binary relation. The axioms of T' are the axioms of T, but with equality replaced by the new relation, together with axioms saying that the new relation is an equivalence relation with infinite classes. Then T' is a complete consistent theory satisfying $\operatorname{acl} A = A$ for every set A of real elements. Hence the lattice of small algebraically closed sets is just the (modular) lattice of small subsets of the big model, and so the relation $A \mathop{\smile\!\!\!\!\!\!|}_C B \iff A \cap B \subseteq C$ is a strict independence relation for T (and agrees with $\mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}$).

Example 1.33. (A theory with two strict independence relations)
Let T be the theory of an equivalence relation with infinitely many infinite classes (i. e., *all* classes are infinite), in the signature of a single relation E. Then both $A \mathop{\smile\!\!\!\!\!\!|}_C B \iff A \cap B \subseteq C$ (thorn-forking for T) and $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{eq}}_C B \iff \operatorname{acl}^{\mathrm{eq}} A \cap \operatorname{acl}^{\mathrm{eq}} B \subseteq \operatorname{acl}^{\mathrm{eq}} C$ (thorn-forking for T^{eq}) define strict independence relations on T, but they are clearly not the same.

Exercises

Exercise 1.34. Check that $A \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_C B$ implies $A \mathop{\smile\!\!\!\!\!\!|}_C B$ in Example 1.31.

Exercise 1.35. Check that $A \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_C B \iff A \cap B \subseteq C$ is a strict independence relation in Example 1.32.

Exercise 1.36. Check that $A \mathop{\smile\!\!\!\!\!\!|}_C B \iff A \cap B \subseteq C$ and $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{eq}}_C B \iff \operatorname{acl}^{\mathrm{eq}} A \cap \operatorname{acl}^{\mathrm{eq}} B \subseteq \operatorname{acl}^{\mathrm{eq}} C$ are strict independence relations in Example 1.33.

Notes

The definition of thorn-forking independence $\mathop{\smile\!\!\!\!\!\!\;|}^{\text{þ}}$ via M-independence $\mathop{\smile\!\!\!\!\!\!\;|}^{\text{M}}$ in Definition 1.26 is new, but M-independence is from [Sch96]. The original motiviation for the definition of $\mathop{\smile\!\!\!\!\!\!\;|}^{\text{M}}$ was of course not thorn-forking. It was Exercise 1.7, which shows that base monotonicity is the only problematic property for $\mathop{\smile\!\!\!\!\!\!\;|}^{\text{a}}$. If we try to force it, we get $\mathop{\smile\!\!\!\!\!\!\;|}^{\text{M}}$. The letter 'M' was chosen because of the notation $\mathrm{M}(x, y)$ for modular pairs in lattices, cf. Exercise 2.41.

Lemma 1.29 is contained in [Ons03a, Lemmas 2.1.2 and 2.1.5]. Alf Onshuus may have overlooked at first the fact (Theorem 1.30, also Theorem 3 in [Sch03]) that thorn-forking is the coarsest strict independence relation in every theory admitting one.

Examples 1.32 and 1.33 are new.

1.6 Just a bunch of silly examples

In Sections 1 and 2 we found some relations that hold between the axioms for independence relations and the existence and symmetry properties. Our aim in this section is to show that we have actually found all of them and, in particular, the axioms for independence relations are independent. Most readers probably want to skip this section.

First we give two examples showing that invariance does not follow from the other axioms:

Example 1.37. (no invariance) Let $\mathop{\smile\!\!\!\!\!\!\;|}$ be any strict independence relation. Let F be a set such that $F \not\subseteq \operatorname{acl} \emptyset$. Define

$$A \mathop{\smile\!\!\!\!\!\!\;|}'_{C} B \iff A \mathop{\smile\!\!\!\!\!\!\;|}_{CF} B.$$

The relation $\mathop{\smile\!\!\!\!\!\!\;|}'$ satisfies all axioms for independence relations except invariance (but not anti-reflexivity). It also satisfies existence and symmetry.

Example 1.38. (no invariance) Consider the theory T from Example 1.33 with its two strict independence relations $\mathop{\smile\!\!\!\!\!\!\;|}$ and $\mathop{\smile\!\!\!\!\!\!\;|}^{\text{eq}}$. Now consider its reduct T' that is just an infinite set without the equivalence relation. Take the big model of T as a big model of T'. Then the relation $\mathop{\smile\!\!\!\!\!\!\;|}^{\text{eq}}$ satisfies all axioms for independence relations with respect to T' except invariance. It also satisfies existence and symmetry.

From this point on, we will only consider invariant relations in this section.

Theorem 1.39. *Consider the following nine axioms that may hold for an invariant relation $\mathop{\smile\!\!\!\!\!\!\mid}$ on the small sets of the big model of a complete theory: monotonicity, base monotonicity, transitivity, normality, extension, finite character, local character, existence, symmetry. The following relations hold between these axioms:*

(1) *An invariant relation satisfying extension and symmetry also satisfies normality.*

(2) *An invariant relation satisfying extension and local character also satisfies existence.*

(3) *An invariant relation satisfying monotonicity, transitivity, normality, existence and symmetry also satisfies extension.*

(4) *An invariant relation satisfying monotonicity, base monotonicity, transitivity, normality, extension, finite character and local character also satisfies symmetry.*

This enumeration is complete: Every relation between these nine axioms that holds in general is a formal consequence of these four relations—with a grain of salt: The question whether monotonicity is needed in (3) is open.

Proof. The relations (1)–(3) hold by Exercise 1.5. Relation (4) is Theorem 1.14.

Completeness of the enumeration is proved by the following series of examples. Examples 1.40, 1.41, 1.42 and 1.43 show that monotonicity, base monotonicity, transitivity and finite character do not follow from any other axioms, respectively. Examples 1.45 and 1.46 show that normality does not follow from any set of other axioms that does not include at least extension and symmetry. Examples 1.48 and 1.47 show that existence does not follow from any set of other axioms that does not include at least extension and local character. Examples 1.49, 1.46, 1.48 and 1.50 show that extension does not follow from any set of other axioms that does not include at least transitivity, normality, existence and symmetry. (Monotonicity is missing from this list.) Examples 1.51, 1.52, 1.53, 1.45, 1.50, 1.54 and 1.55 show that symmetry does not follow from any set of other axioms that does not include at least monotonicity, base monotonicity, transitivity, normality, extension, finite character and local character.

The author could not find an example satisfying all axioms except extension and monotonicity. □

Example 1.40. (no monotonicity)
Consider the theory from Example 1.33 with its two strict independence relations $\mathop{\smile\!\!\!\!\!\!\mid}$ and $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{eq}}$. Define

$$A \mathop{\smile\!\!\!\!\!\!\mid}'_C B \iff \begin{cases} A \mathop{\smile\!\!\!\!\!\!\mid}_C B \text{ if } A \text{ and } B \text{ are infinite} \\ A \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{eq}}_C B \text{ if } A \text{ or } B \text{ is finite.} \end{cases}$$

The relation $\mathop{\smile\!\!\!\!\!\!\mid}'$ satisfies all axioms for strict independence relations except monotonicity. It also satisfies existence and symmetry.

Example 1.41. (no base monotonicity)
Consider the theory from Examples 1.4 and 1.31. The relation $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{a}}$ from Exercise 1.7 satisfies all axioms for strict independence relations except base monotonicity. It also satisfies existence and symmetry.

Example 1.42. (no transitivity)
Consider the theory from Example 1.33 with its two strict independence relations $\mathop{\smile\!\!\!\!\!\!\mid}$ and $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{eq}}$. Define

$$A \mathop{\smile\!\!\!\!\!\!\mid}'_C B \iff \begin{cases} A \mathop{\smile\!\!\!\!\!\!\mid}_C B \text{ if } C \text{ is infinite} \\ A \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{eq}}_C B \text{ if } C \text{ is finite.} \end{cases}$$

The relation $\mathop{\smile\!\!\!\!\!\!\mid}'$ satisfies all axioms for strict independence relations except transitivity. It also satisfies existence and symmetry. (By interchanging 'finite' and 'infinite' we would get another example without base monotonicity.)

Example 1.43. (no finite character)
In the theory of an infinite set with no structure, consider the relation

$$A \mathop{\smile\!\!\!\!\!\!\mid}_C B \iff \big|(A \cap B) \setminus C\big| \leq \aleph_0.$$

The relation $\mathop{\smile\!\!\!\!\!\!\mid}$ satisfies all axioms for independence relations except finite character. It also satisfies existence and symmetry (but not anti-reflexivity).

To get anti-reflexivity as well, consider the theory of an equivalence relation E with infinitely many infinite classes. Let π be the obvious projection from the standard sort to the imaginary sort of equivalence classes of E. Define

$$A \mathop{\smile\!\!\!\!\!\!\mid}'_C B \iff A \cap B \subseteq C \text{ and } \big|\big(\pi(A) \cap \pi(B)\big) \setminus \pi(C)\big| \leq \aleph_0.$$

The relation $\mathop{\smile\!\!\!\!\!\!\mid}'$ satisfies all axioms for strict independence relations except finite character. It also satisfies existence and symmetry.

Example 1.44. (no local character)
Consider the theory of the random graph, i. e., the Fraïssé limit of the finite undirected graphs, with the following relation:

$$A \underset{C}{\downarrow} B \iff A \cap B \subseteq C \textit{ and there is no edge from } A \setminus C \textit{ to } B \setminus C.$$

The relation $\downarrow$ satisfies all axioms for strict independence relations except local character. It also satisfies existence and symmetry.

Example 1.45. (no normality, no symmetry)
Consider the following relation:

$$A \underset{C}{\downarrow} B \iff \operatorname{acl} A \cap \operatorname{acl}(BC) \subseteq \operatorname{acl} C.$$

It always satisfies all axioms for strict independence relations other than normality, and it also satisfies existence for every theory. But in the theory from Example 1.4 let $a \neq c$, and b be points such that there is an edge from a to b and from b to c. Then $a \downarrow_c b$, $ac \not\downarrow_c b$ and $b \not\downarrow_c a$, so $\downarrow$ does not satisfy normality or symmetry.

Example 1.46. (no normality, no extension)
Consider the following relation:

$$A \underset{C}{\downarrow} B \iff \operatorname{acl} A \cap \operatorname{acl} B \subseteq \operatorname{acl} C.$$

It always satisfies all axioms for strict independence relations other than normality, and it also satisfies symmetry for every theory. But in the theory from Example 1.4 let $a \neq c$, and b be points such that there is an edge from a to b and from b to c. Then $a \downarrow_c b$ but $ac \not\downarrow_c b$, so $\downarrow$ does not satisfy normality. It easily follows that $\downarrow$ does not satisfy extension either.

Example 1.47. (no local character, no existence)
The empty ternary relation satisfies all axioms for strict independence relations except local character. It also satisfies symmetry, but not existence.

Example 1.48. (no extension, no existence)
Let $\downarrow$ be a strict independence relation for some theory T. Define $\downarrow'$ as follows:

$$A \underset{C}{\downarrow'} B \quad\iff\quad \big(|C| \geq \aleph_0 \textit{ and } A \underset{C}{\downarrow} B\big) \textit{ or } A \subseteq C \textit{ or } B \subseteq C.$$

The relation $\downarrow'$ satisfies all axioms for strict independence relations except extension. It also satisfies symmetry, but not existence.

Example 1.49. (no extension, no transitivity)
Given any strict independence relation $\mathop{\smile\!\!\!\!\!\!|}\,$, consider the following relation:

$$A \mathop{\smile\!\!\!\!\!\!|'}_{C} B \iff A \mathop{\smile\!\!\!\!\!\!|}_{C} B \text{ or } \big(|A \setminus C| \leq 1 \textit{ and } |B \setminus C| \leq 1\big).$$

It satisfies all axioms for strict independence relations except transitivity and extension, and it also satisfies existence and symmetry.

Example 1.50. (no extension, no symmetry)
Consider the theory from Examples 1.4 and 1.31. By Lemma 1.29 the relation $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}}$ satisfies the axioms of strict independence relations except extension and local character. $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}}$ also satisfies local character and existence because $\mathop{\smile\!\!\!\!\!\!|}^{\flat}$ does. We have already seen that $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}}$ does not satisfy extension.

$\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}}$ is not symmetric either: Suppose b lies between a and c. Then $a \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}} bc$ as we have just seen. But it is easy to see that $bc \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}} a$.

Example 1.51. (no monotonicity, no symmetry)
For any strict independence relation $\mathop{\smile\!\!\!\!\!\!|}$ consider the following relation:

$$A \mathop{\smile\!\!\!\!\!\!|'}_{C} B \iff A \mathop{\smile\!\!\!\!\!\!|}_{C} B \textit{ or } |A \setminus \operatorname{acl} C| \geq 2.$$

The relation $\mathop{\smile\!\!\!\!\!\!|'}$ satisfies all axioms for independence relations except monotonicity. It also satisfies extension, but not symmetry.

Example 1.52. (no base monotonicity, no symmetry)
Consider the theory of dense linear orders with the following relation:

$$A \mathop{\smile\!\!\!\!\!\!|}_{C} B \iff A \cap B \subseteq C \textit{ or } \exists a \in A \exists c \in C : a < c.$$

The relation $\mathop{\smile\!\!\!\!\!\!|}$ satisfies all axioms for independence relations except base monotonicity. It also satisfies existence, but not symmetry.

Example 1.53. (no transitivity, no symmetry)
For any independence relation $\mathop{\smile\!\!\!\!\!\!|}$ consider the following relation:

$$A \mathop{\smile\!\!\!\!\!\!|'}_{C} B \iff A \mathop{\smile\!\!\!\!\!\!|}_{C} B \textit{ or } |A \setminus C| \leq 1.$$

The relation $\mathop{\smile\!\!\!\!\!\!|'}$ satisfies all axioms for independence relations except transitivity. It also satisfies existence, but not symmetry.

Example 1.54. (no finite character, no symmetry)
Given any strict independence relation $\mathop{\smile\hskip-0.9em\raise0.5ex\hbox{$|$}}$, consider the following relation:

$$A \mathop{\smile\hskip-0.9em\raise0.5ex\hbox{$|$}}'_C B \iff \exists \text{ finite } B_0 \subseteq B \text{ such that } A \mathop{\smile\hskip-0.9em\raise0.5ex\hbox{$|$}}_{B_0 C} C.$$

The relation $\mathop{\smile\hskip-0.9em\raise0.5ex\hbox{$|$}}'$ always satisfies all axioms for strict independence relations except finite character, which it does not satisfy. It also satisfies existence, but not necessarily symmetry.

Now let T be the theory of ω cross-cutting equivalence relations ϵ_i with infinitely many classes each. Let $\mathop{\smile\hskip-0.9em\raise0.5ex\hbox{$|$}} = \mathop{\smile\hskip-0.9em\raise0.5ex\hbox{$|$}}^{\mathrm{f}}$. Let $(a_i)_{i<\omega}$ and b be such that $\models \epsilon_i(a_j, b) \iff i = j$. Then $(a_i)_{i<\omega} \mathop{\smile\hskip-0.9em\raise0.5ex\hbox{$|$}}'_\emptyset b$ and $b \not\mathop{\smile\hskip-0.9em\raise0.5ex\hbox{$|$}}'_\emptyset (a_i)_{i<\omega}$.

Example 1.55. (no local character, no symmetry)
We will extend the theories T_0 and T from Example 1.15. First we describe the signature of the respective extensions T_0^* and T^*: It has the sorts P ('points') and E ('equivalence relations') as well as a new sort Γ ('equivalence classes'). The functions and relations of T_0^* consist of the relation $p \sim_e q$ (for $p, q \in P$ and $e \in E$), a new relation written (slightly abusing notation) as $p/e = c$ for $p \in P$, $e \in E$ and $c \in \Gamma$, and a function $\epsilon : \Gamma \to E$.

The axioms of T_0^* include those of T_0, i. e., $\sim_e$ is an equivalence relation for every $e \in E$. They also say that $\exists_{\leq 1} c(p/e = c)$, so it makes sense to regard p/e as a partial function $P \times E \to \Gamma$ which we will use informally in the following. The other axioms say $\epsilon(p/e) = e$ (if p/e exists) and $p \sim_e q \leftrightarrow p/e = q/e$ (also if p/e exists).

Clearly every model of T_0 is also a model of T_0^*, and if we restrict a model of T_0^* to the sorts P and E we get a model of T_0. By the same arguments as for T_0 we can find an ω-categorical theory T^* with elimination of quantifiers which is the Fraïssé limit of the finite models of T_0^*. So T^* extends both T and T_0^*.

For any subset A of the big model of T^* we write $P(A) = A \cap P$, $E(A) = (A \cap E) \cup \epsilon(A \cap \Gamma)$ and $\Gamma(A) = (A \cap \Gamma) \cup \{p/e \mid p \in P(A), e \in E(A)\}$. It is not hard to check that $\operatorname{acl} A = \operatorname{dcl} A = P(A) \cup E(A) \cup \Gamma(A)$. It easily follows that

$$A \mathop{\smile\hskip-0.9em\raise0.5ex\hbox{$|$}}^{\mathrm{M}}_C B \iff \left(\begin{array}{c} P(A) \cap P(B) \subseteq P(C) \text{ and} \\ E(A) \cap E(B) \subseteq E(C) \text{ and} \\ P(A)/e \cap P(B)/e \subseteq \Gamma(C) \text{ for all } e \in E(BC) \end{array} \right).$$

Using this, it is not hard to check that $A \mathop{\smile\hskip-0.9em\raise0.5ex\hbox{$|$}}^{\mathrm{M}}_C B \implies A \mathop{\smile\hskip-0.9em\raise0.5ex\hbox{$|$}}^{\flat}_C B$ and that $A \mathop{\smile\hskip-0.9em\raise0.5ex\hbox{$|$}}^{\mathrm{M}}_C B \implies A \mathop{\smile\hskip-0.9em\raise0.5ex\hbox{$|$}}^{\mathrm{d}}_C B$, from which it easily follows that $\mathop{\smile\hskip-0.9em\raise0.5ex\hbox{$|$}}^{\mathrm{M}} = \mathop{\smile\hskip-0.9em\raise0.5ex\hbox{$|$}}^{\flat} = \mathop{\smile\hskip-0.9em\raise0.5ex\hbox{$|$}}^{\mathrm{d}} = \mathop{\smile\hskip-0.9em\raise0.5ex\hbox{$|$}}^{\mathrm{f}}$. Hence $\mathop{\smile\hskip-0.9em\raise0.5ex\hbox{$|$}}^{\mathrm{f}}$ satisfies all axioms for strict independence relations except local character (which would imply that there is a strict independence relation for T^*). It also satisfies existence, though not symmetry.

Exercises

Exercise 1.56.
Show that $\underset{\smile}{|}'$ in Example 1.48 has the stated properties.

Notes

The author wishes to excuse for all the nonsensical examples in this section. Once he had found the first few, he could not resist the temptation to do a systematic search, the findings of which are now dumped on the reader.

Chapter 2

Forking

In this chapter we improve part of Chapter 1 by exploring part of the local (i. e., relating to formulas), or combinatorial, foundation of forking theory. We will introduce the concept of inconsistency pairs. If Ω is a set of inconsistency pairs we get a relation $\mathop{\smile\hskip-0.9em ^|\ }^{\Omega}$ such that $\mathop{\smile\hskip-0.9em ^|\ }^{\Omega *}$ is a good candidate for being an independence relation. In particular, $\mathop{\smile\hskip-0.9em ^|\ }^{\Omega *} = \mathop{\smile\hskip-0.9em ^|\ }^{\mathrm{f}}$ or $\mathop{\smile\hskip-0.9em ^|\ }^{\Omega *} = \mathop{\smile\hskip-0.9em ^|\ }^{\text{þ}}$ for suitable choices of Ω. This will allow us to find out more about $\mathop{\smile\hskip-0.9em ^|\ }^{\mathrm{f}}$ and $\mathop{\smile\hskip-0.9em ^|\ }^{\text{þ}}$.

As in the previous chapter, the exposition is essentially self-contained. I write $\bar{a}_{<k}$ for the tuple $\bar{a}_0\bar{a}_1\ldots\bar{a}_{k-1}$ and $\bar{a}_{<\omega}$ for the sequence $(\bar{a}_i)_{i<\omega}$. Tuples of variables or elements are often implicitly assumed to be compatible: of the same lengths and with the same sorts at corresponding positions.

2.1 Strong finite character

The purpose of this section is to give a foretaste of the improved results which we will get in this chapter, while postponing the technicalities of the next two sections as long as possible. Here we show that we could have had slightly stronger results in Chapter 1 if we had required the following stronger condition instead of finite character:

Definition 2.1. *The* strong finite character *condition is the following strong variant of the finite character axiom:*

(strong finite character)

If $A \not\mathop{\smile\hskip-0.9em ^|\ }_C B$, then there are finite tuples $\bar{a} \in A$, $\bar{b} \in B$ and $\bar{c} \in C$ and a formula $\varphi(\bar{x}, \bar{y}, \bar{z})$ without parameters such that

- *$\models \varphi(\bar{a}, \bar{b}, \bar{c})$, and*
- *$\bar{a}' \not\mathop{\smile\hskip-0.9em ^|\ }_C \bar{b}$ for all $\bar{a}'$ satisfying $\models \varphi(\bar{a}', \bar{b}, \bar{c})$.*

As we will see, this new condition is satisfied by all relations that are of interest to us. It has two properties that make it more convenient than finite character. The first is the following supplement to Lemma 1.17 that could have spared us the somewhat contrived proof of Theorem 1.18.

Lemma 2.2.
Let $\mathop{\smile\!\!\!\!\!\mid}$ be a relation that satisfies invariance, monotonicity and the strong finite character condition. Then $\mathop{\smile\!\!\!\!\!\mid}^*$ also satisfies the strong finite character condition.

Proof. Suppose $\bar{a} \not\mathop{\smile\!\!\!\!\!\mid}^*_C B$ ($\bar{a}$ being a sequence of arbitrary length), and let this be witnessed by $\hat{B} \supseteq B$ such that $\bar{a}' \not\mathop{\smile\!\!\!\!\!\mid}^*_C \hat{B}$ for all $\bar{a}' \equiv_{BC} \bar{a}$. Let $\bar{x}$ be a sequence of the same length as $\bar{a}$, and let $p(\bar{x})$ be the set of formulas over $\hat{B}C$ consisting of the negations of all those formulas $\varphi_i(\bar{x}, \bar{b}_i, \bar{c}_i)$ with parameters $\bar{b}_i \in \hat{B}$ and $\bar{c}_i \in C$ that have the property that $\bar{a}' \not\mathop{\smile\!\!\!\!\!\mid}_C \bar{b}_i$ for all $\bar{a}'$ satisfying $\models \varphi_i(\bar{a}', \bar{b}_i, \bar{c}_i)$. By choice of $\hat{B}$ and strong finite character of $\mathop{\smile\!\!\!\!\!\mid}$, $p(\bar{x}) \cup \mathrm{tp}(\bar{a}/BC)$ is inconsistent. So by compactness there is a formula $\psi(\bar{x}, \bar{b}, \bar{c}) \in \mathrm{tp}(\bar{a}/BC)$ such that $p(\bar{x}) \cup \{\psi(\bar{x}, \bar{b}, \bar{c})\}$ is inconsistent.

Now suppose $\bar{a}'$ satisfies $\models \psi(\bar{a}', \bar{b}, \bar{c})$. To finish our proof we claim that $\bar{a}' \not\mathop{\smile\!\!\!\!\!\mid}^*_C \bar{b}$. Otherwise there would be $\bar{a}^* \equiv_{C\bar{b}} \bar{a}'$ such that $\bar{a}^* \mathop{\smile\!\!\!\!\!\mid}_C \hat{B}$. But then $\models \psi(\bar{a}^*, \bar{b}, \bar{c})$ would also hold. On the other hand, $\bar{a}^*$ would realise $p(\bar{x})$, in contradiction to inconsistency of $p(\bar{x}) \cup \{\psi(\bar{x}, \bar{b}, \bar{c})\}$. $\square$

For the second advantage of strong finite character recall that a type $p(\bar{x})$ is called finitely satisfied in a set C if for every formula $\varphi(\bar{x}, \bar{b}) \in p$ there is a tuple $\bar{c} \in C$ such that $\models \varphi(\bar{c}, \bar{b})$.

Remark 2.3. *Suppose $\mathop{\smile\!\!\!\!\!\mid}$ satisfies monotonicity and strong finite character, and $\bar{a}, B, C$ are such that $C \mathop{\smile\!\!\!\!\!\mid}_C B$ holds and $\mathrm{tp}(\bar{a}/BC)$ is finitely satisfied in C. Then $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_C B$.*

Proof. Suppose $\bar{a} \not\mathop{\smile\!\!\!\!\!\mid}_C B$. Let $\varphi(\bar{x}_0, \bar{y}, \bar{z})$ and $\bar{a}_0 \subseteq \bar{a}$, $\bar{b} \in B$, $\bar{c} \in C$ be as in the strong finite character condition. Since $\mathrm{tp}(\bar{a}/BC)$ is finitely satisfied in C there is $\bar{a}' \in C$ such that $\models \varphi(\bar{a}', \bar{b}, \bar{c})$ holds. Hence $\bar{a}' \not\mathop{\smile\!\!\!\!\!\mid}_C B$, hence $C \not\mathop{\smile\!\!\!\!\!\mid}_C B$ by monotonicity. $\square$

This is quite useful because of the following well-known fact:

Remark 2.4. *For any $\bar{a}$, B there is a subset $C \subseteq \bar{a}$ of size $|C| \leq |T| + |B|$ such that $\mathrm{tp}(\bar{a}/BC)$ is finitely satisfied in C.*

Proof. Let $C_0 = \emptyset$. Given any set C_n we construct $C_{n+1} \supseteq C_n$ as follows: For every formula $\varphi(\bar{x}_0, \bar{b})$, $\bar{b} \in BC_n$, that is satisfied by a finite subtuple $\bar{a}_0 \subseteq \bar{a}$, we make sure that C_{n+1} contains one such tuple $\bar{a}_0$. Clearly we can make sure that $|C_{n+1}| \leq |T| + |C_n|$. Now we can just take $C = \bigcup_{n<\omega} C_n$. $\square$

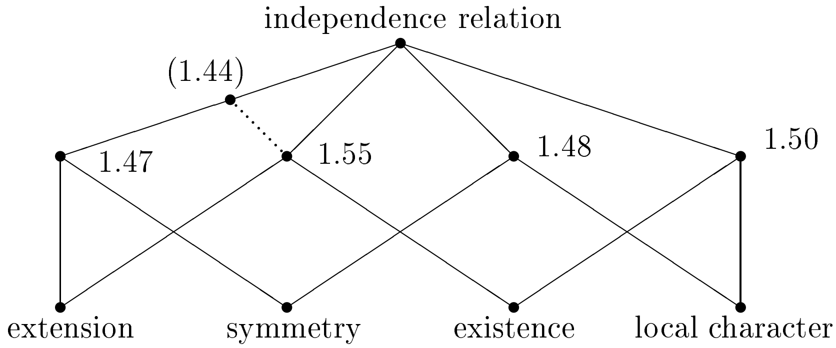

Figure 2.1: Classification of relations satisfying the first 5 axioms of independence relations and finite character, according to which of 4 remaining properties hold. For each point in the middle row of this lattice diagram there is an example in Section 1.6. These can be used to assemble examples for the bottom row. If we also require strong finite character, the dotted line and the point represented by Example 1.44 disappear.

Putting both results together it is easy to get the dual (left and right sides reversed) of local character. Therefore we have:

Theorem 2.5. *Suppose $\mathop{\smile\!\!\!\!\!\!\!\mid}$ satisfies the first five axioms for independence relations as well as the strong finite character condition. Then $\mathop{\smile\!\!\!\!\!\!\!\mid}$ is an independence relation if and only if $\mathop{\smile\!\!\!\!\!\!\!\mid}$ satisfies existence and symmetry.*

Proof. First note that strong finite character implies finite character. We already know the forward direction, so we only need to prove extension and local character from existence and symmetry.

Extension easily follows from transitivity, normality, existence and symmetry. For local character we can take $\kappa(B) = (|T| + |B|)^+$: Given $\bar{a}$ and B there is $C \subseteq \bar{a}$ such that $|C| < \kappa(B)$ and $\mathrm{tp}(\bar{a}/BC)$ is finitely satisfied in C. Now $C \mathop{\smile\!\!\!\!\!\!\!\mid}_C B$ holds by existence, so $\bar{a} \mathop{\smile\!\!\!\!\!\!\!\mid}_C B$ by monotonicity and strong finite character. □

It follows that the relation in Example 1.44 does not have strong finite character.

Exercises

Exercise 2.6. ($\mathop{\smile\!\!\!\!\!\!\!\mid}^{\mathrm{M}}$ has strong finite character)

The relation $\mathop{\smile\!\!\!\!\!\!\!\mid}^{\mathrm{M}}$ always satisfies the strong finite character condition.

Exercise 2.7. (Figure 2.1)

Check that the examples mentioned in the middle row of Figure 2.1 satisfy the strong finite character condition. For Example 1.48 assume that $\mathop{\smile\!\!\!\!\!\!\!\mid}$ satisfies strong finite character.

Exercise 2.8. (alternative definition for strong finite character)

Suppose the relation $\mathop{\smile\!\!\!\!\!\mid}$ satisfies invariance, monotonicity and extension. Prove that it satisfies the strong finite character condition if and only if it satisfies the following condition: For any sequence of variables $\bar{x}$ and any sets B, C, the set $\{\operatorname{tp}(\bar{a}/BC) \mid \bar{a} \mathop{\smile\!\!\!\!\!\mid}_C B\}$ is a closed subset of $\mathrm{S}^{\bar{x}}(BC)$.

Notes

The term 'strong finite character' is probably new, but the property itself is essentially the anonymous axiom A.7 in [Mak84]. Exercise 2.8 (read in conjunction with Theorem 2.5) shows that independence relations satisfying the strong finite character condition are precisely the relations considered in [Cas03, Section 3]. None of the arguments in this section is new.

I do not believe that *every* independence relation has strong finite character, but I do not have a counter-example (cf. Question A.3).

2.2 Local dividing

Definition 2.9. *The formula $\psi(\bar{y}_{<k})$ is called a* k-inconsistency witness *for $\varphi(\bar{x};\bar{y})$ if the formula $\left(\bigwedge_{i<k} \varphi(\bar{x};\bar{y}_i)\right) \wedge \psi(\bar{y}_{<k})$ is inconsistent. When the precise value of k is immaterial we will omit it. We write*

$$\Xi = \left\{ \left(\varphi(\bar{x};\bar{y}), \psi(\bar{y}_{<k})\right) \;\middle|\; \psi \textit{ is a } k\textit{-inconsistency witness for } \varphi;\, k < \omega \right\}$$

for the set of all inconsistency pairs.

Note that in the preceding definition the free variables of $\varphi(\bar{x};\bar{y})$ are partitioned in two blocks. The definition depends crucially on this partition.

A k-inconsistency witness $\psi(\bar{y}_{<k})$ for $\varphi(\bar{x};\bar{y})$ 'witnesses' k-inconsistency in the following way: Suppose $(\bar{b}_i)_{i<\omega}$ is a sequence such that $\models \psi(\bar{b}_{i_0}, \dots, \bar{b}_{i_{k-1}})$ for any $i_0 < \cdots < i_{k-1} < \omega$. Then the set $\{\varphi(\bar{x};\bar{b}_i) \mid i < \omega\}$ is k-inconsistent, i. e., there is no tuple $\bar{a}$ satisfying k formulas from the set simultaneously.

Definition 2.10. *A formula $\varphi(\bar{x};\bar{b})$* (φ,ψ)-divides *over a set C if $(\varphi,\psi) \in \Xi$ and there is a sequence $\bar{b}_{<\omega}$ such that*

- *each $\bar{b}_i$ realises $\operatorname{tp}(\bar{b}/C)$, and*
- *$\models \psi(\bar{b}_{i_0}, \dots, \bar{b}_{i_{k-1}})$ holds for all $i_0 < \ldots < i_{k-1} < \omega$.*

We say that $\bar{b}_{<\omega}$ witnesses *that $\varphi(\bar{x};\bar{b})$ (φ,ψ)-divides over C.*

A partial type $p(\bar{x})$ (φ,ψ)-divides *over a set C if it contains a formula $\varphi(\bar{x};\bar{b}) \in p(\bar{x})$ that (φ,ψ)-divides over C.*

Note that when $\varphi(\bar{x};\bar{b})$ (φ,ψ)-divides over a set C, then there is a sequence $\bar{b}_{<\omega}$ witnessing this with $\bar{b}_0 = \bar{b}$. Also note that $\varphi(\bar{x};\bar{b})$ also (φ,ψ)-divides over every subset of C.

Definition 2.11. *Let $\Omega \subseteq \Xi$ be a subset of Ξ that is* closed under variable substitution *in the following sense: If $\big(\varphi(\bar{x};\bar{y}),(\psi(\bar{y}_{<k})\big) \in \Omega$ and $\bar{u}$, $\bar{v}$, $\bar{v}_{<k}$ are appropriate tuples of variables—possibly with repetitions, but $\bar{u}$, $\bar{v}$ and the tuples $\bar{v}_i$ being pairwise disjoint from each other—then $\big(\varphi(\bar{u},\bar{v}),\psi(\bar{v}_{<k})\big) \in \Omega$.*

We say that a partial type $p(\bar{x})$ Ω-divides *over a set C if it (φ,ψ)-divides over C for some $(\varphi,\psi) \in \Omega$. We define a relation $\mathop{\smile\!\!\!\!\!\!|}^{\Omega}$ as follows:*

$$A \mathop{\smile\!\!\!\!\!\!|}^{\Omega}_{C} B \iff \textit{there is no } \bar{a} \in A \textit{ such that } \operatorname{tp}(\bar{a}/BC)\ \Omega\textit{-divides over } C.$$

Note that Ξ itself is closed under variable substitution.

Proposition 2.12. *Suppose $\Omega \subseteq \Xi$ is closed under variable substitution. Then $\mathop{\smile\!\!\!\!\!\!|}^{\Omega}$ satisfies the following axioms for independence relations: invariance, monotonicity, base monotonicity and finite character. In fact, $\mathop{\smile\!\!\!\!\!\!|}^{\Omega}$ has strong finite character. Moreover, $A \mathop{\smile\!\!\!\!\!\!|}^{\Omega}_{B} B$ and $A \mathop{\smile\!\!\!\!\!\!|}^{\Omega}_{A} B$ for any sets A and B.*

Proof. Invariance and monotonicity are obvious.

Base monotonicity: Suppose $A \not\mathop{\smile\!\!\!\!\!\!|}^{\Omega}_{C} B$ and $D \subseteq C \subseteq B$. It suffices to show that $A \not\mathop{\smile\!\!\!\!\!\!|}^{\Omega}_{D} B$. There is $\bar{a} \in A$, $(\varphi,\psi) \in \Omega$ and $\bar{b} \in B$ such that $\varphi(\bar{x};\bar{b}) \in \operatorname{tp}(\bar{a}/B)$ and $\varphi(\bar{x};\bar{b})$ (φ,ψ)-divides over C. It is immediate from the definition of (φ,ψ)-dividing that $\varphi(\bar{x};\bar{b})$ also (φ,ψ)-divides over D. So $A \not\mathop{\smile\!\!\!\!\!\!|}^{\Omega}_{D} B$ does in fact hold.

Strong finite character: Suppose $A \not\mathop{\smile\!\!\!\!\!\!|}^{\Omega}_{C} B$. Let $\bar{a} \in A$ be such that $\operatorname{tp}(\bar{a}/BC)$ Ω-divides over C. So there is $(\varphi,\psi) \in \Omega$ and $\bar{b} \in BC$ such that $\varphi(\bar{x};\bar{b}) \in \operatorname{tp}(\bar{a}/BC)$ and $\varphi(\bar{x};\bar{b})$ (φ,ψ)-divides over C. Hence for every $\bar{a}'$ satisfying $\models \varphi(\bar{a}';\bar{b})$, $\operatorname{tp}(\bar{a}'/\bar{b}C)$ also (φ,ψ)-divides over C, so $\bar{a}' \not\mathop{\smile\!\!\!\!\!\!|}^{\Omega}_{C} \bar{b}$.

For the first 'moreover' statement, suppose $(\varphi,\psi) \in \Xi$, $\models \varphi(\bar{a};\bar{b})$ for some tuples $\bar{a} \in A$ and $\bar{b} \in B$, and $\varphi(\bar{x};\bar{b})$ (φ,ψ)-divides over B. This would be witnessed by a sequence $\bar{b}_{<\omega}$ of tuples realising $\operatorname{tp}(\bar{b}/B)$, so $\bar{b}_i = \bar{b}$. But then $\models \big(\bigwedge_{i<k}\varphi(\bar{a};\bar{b}_i)\big) \wedge \psi(\bar{b}_{<k})$, contradicting the assumption that (φ,ψ) is a k-inconsistency witness.

For the second 'moreover' statement, suppose $(\varphi,\psi) \in \Xi$, $\models \varphi(\bar{a};\bar{b})$ for some tuples $\bar{a} \in A$ and $\bar{b} \in B$, and $\varphi(\bar{x};\bar{b})$ (φ,ψ)-divides over A. This would be witnessed by a sequence $\bar{b}_{<\omega}$ of tuples realising $\operatorname{tp}(\bar{b}/A)$. But then again $\models \big(\bigwedge_{i<k}\psi(\bar{a};\bar{b}_i)\big) \wedge \psi(\bar{b}_{<k})$, contradicting the assumption that (φ,ψ) is a k-inconsistency witness. ∎

So the missing axioms are transitivity, normality, extension and local character. Heuristically speaking, from our experience in Chapter 1 we can say that extension is probably no problem since we can fix it by passing to $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}^{\Omega *}$, while we could not expect local character to hold in general. Only the fact that we cannot prove transitivity and normality is a bit annoying (since both are among the first five axioms), so let us check that the problem is real:

Example 2.13. Let T be the theory of an infinite set in the empty signature. Let Ω consist of all inconsistency pairs of the form $\big(\varphi(xx';yy'), \psi(y_0y_0', y_1y_1')\big)$, where $\varphi(xx';yy') \equiv x \neq x' \wedge x = y \wedge x' = y'$ and $\psi(y_0y_0', y_1y_1') \equiv y_0 \neq y_1 \wedge y_0' \neq y_1'$.

It is not hard to see that $A \mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}^{\Omega}_{C} B \iff \big|(A \cap B) \setminus C\big| \leq 1$, that $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}^{\Omega}$ satisfies extension, local character, existence and symmetry, and that $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}^{\Omega} = \mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}^{\Omega *}$. But $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}^{\Omega}$ does not satisfy transitivity: Suppose $b \neq c$. Then $bc \mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}^{\Omega}_{c} bc$ and $c \mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}^{\Omega}_{\emptyset} bc$, but $bc \mathop{\not\smile\hspace{-0.9em}\mid\hspace{0.6em}}^{\Omega}_{\emptyset} bc$. Since $b \mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}^{\Omega}_{\emptyset} bc$ and $bc \mathop{\not\smile\hspace{-0.9em}\mid\hspace{0.6em}}^{\Omega}_{\emptyset} bc$, $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}^{\Omega}$ does not satisfy normality, either.

Example 2.14. Let T be a theory in which there is a type that forks over its domain in the sense of Shelah-forking. Two examples of this phenomenon were given by Saharon Shelah in [She90, Exercise III.1.3]. In Proposition 2.31 below we will see that $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}^{\Xi} = \mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}^{\mathrm{d}}$. From this it easily follows that $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}^{\Xi}$ does not satisfy extension or existence. Moreover, it follows from Theorem 2.32 below that $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}^{\Xi} = \mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}^{\mathrm{d}}$ does not satisfy local character or symmetry, either.

The choice of Ω in Example 2.13 was of course perverse. Here are two natural conditions that we may require so that $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}^{\Omega}$ makes sense:

Definition 2.15. *Suppose $\Omega \subseteq \Xi$ is closed under variable substitution.*

We say that Ω is transitive *if the following holds:*
Suppose $(\varphi(\bar{y};\bar{x}), \psi(\bar{x}_{<k}) \in \Omega$ and $C \mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}^{\Omega}_{D} \bar{a}$, where $D \subseteq C$. If $\varphi(\bar{y};\bar{a})$ (φ,ψ)-divides over D then $\varphi(\bar{y};\bar{a})$ (φ,ψ)-divides over C.

We say that Ω is normal *if the following holds:*
If $\big(\varphi(\bar{x},\bar{z};\bar{y}), \psi(\bar{y}_{<k})\big) \in \Omega$, then also $\big(\varphi(\bar{x};\bar{z},\bar{y}), \psi'(\bar{y}_{<k},\bar{z}_{<k})\big) \in \Omega$, where $\psi'(\bar{y}_{<k},\bar{z}_{<k}) \equiv \psi(\bar{y}_{<k}) \wedge (z_0 = z_1 = \cdots = z_{k-1})$.

Proposition 2.16.

(1) *If Ω is transitive, then $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}^{\Omega}$ satisfies the transitivity axiom.*

(2) *If Ω is normal, then $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}^{\Omega}$ satisfies the normality axiom.*

Proof. (1) Suppose Ω is transitive, $D \subseteq C \subseteq B$, $C \mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}^{\Omega}_{D} A$ and $B \mathop{\not\smile\hspace{-0.9em}\mid\hspace{0.6em}}^{\Omega}_{D} A$. Then there is $(\varphi,\psi) \in \Omega$, $\bar{b} \in B$ and $\bar{a} \in AD$ such that $\models \varphi(\bar{b};\bar{a})$ and $\varphi(\bar{y};\bar{a})$

(φ, ψ)-divides over D. Since $C \mathop{\smile\hspace{-0.9em}\mid}_D \bar{a}$ it follows that $\varphi(\bar{y}; \bar{a})$ also (φ, ψ)-divides over C. Hence $B \mathop{\not\smile\hspace{-0.9em}\mid}^{\Omega}_C A$.

(2) Now suppose instead that Ω is normal and $AC \mathop{\not\smile\hspace{-0.9em}\mid}^{\Omega}_C B$. Then there is $\big(\varphi(\bar{x}, \bar{z}; \bar{y}), \psi(\bar{x}_{<k}, \bar{z}_{<k})\big) \in \Omega$, $\bar{a} \in A$, $\bar{b} \in B$ and $\bar{c} \in C$ such that $\models \varphi(\bar{a}, \bar{c}; \bar{b})$ and $\varphi(\bar{x}, \bar{z}; \bar{b})$ (φ, ψ)-divides over C. Let ψ' be as in the definition of normality for Ω. Then $\big(\varphi(\bar{x}; \bar{z}, \bar{b}), \psi'\big) \in \Omega$ and $\varphi(\bar{x}; \bar{c}, \bar{b})$ clearly (φ, ψ')-divides over C. Hence $A \mathop{\not\smile\hspace{-0.9em}\mid}^{\Omega}_C B$. $\square$

Exercises

Exercise 2.17. (Δ-forking)
For $\Omega \subseteq \Xi$ let $\Omega{\restriction}\bar{x} = \big\{\big(\varphi(\bar{x}; \bar{y}), \psi(\bar{y}_{<k})\big) \;\big|\; k < \omega,\; (\varphi, \psi) \in \Omega\big\}$ be the set of those tuples $(\varphi, \psi) \in \Omega$ for which the left block of variables of φ is $\bar{x}$. For any $\Delta \subseteq \Xi{\restriction}\bar{x}$, we say that a partial type $p(\bar{x})$ *Δ-forks* over a set C if there are $n < \omega$, $\big(\varphi^i(\bar{x}; \bar{y}^i), \psi^i(\bar{y}^i_{<k_i})\big) \in \Delta$ for $i < n$, and tuples $\bar{b}^0, \ldots, \bar{b}^{n-1}$ such that $p(\bar{x}) \vdash \bigvee_{i<n} \varphi^i(\bar{x}; \bar{b}^i)$ and $\varphi^i(\bar{x}; \bar{b}^i)$ (φ^i, ψ^i)-divides over C for each $i < n$.

Suppose $\Omega \subseteq \Xi$ is closed under variable substitution. Show that $\bar{a} \mathop{\smile\hspace{-0.9em}\mid}^{\Omega*}_C B$ iff $\operatorname{tp}(\bar{a}/BC)$ does not Δ-fork over C for any finite $\Delta \subseteq \Omega$.

Exercise 2.18. (more on dividing and forking of formulas, cf. Exercise 1.24)
(i) Given any formula $\varphi(\bar{x}; \bar{b})$, show that $\varphi(\bar{x}; \bar{b})$ divides over C iff $\varphi(\bar{x}; \bar{b})$ (φ, ψ)-divides over C for some formula $\psi(\bar{x}_{<k})$ that is a k-inconsistency witness for $\varphi(\bar{x}; \bar{y})$.
(ii) Show that $\varphi(\bar{x}; \bar{b})$ forks over C iff $\varphi(\bar{x}; \bar{b})$ Δ-forks over C for some set $\Delta \subseteq \Xi{\restriction}\bar{x}$.

Exercise 2.19.
Check the claim that $A \mathop{\smile\hspace{-0.9em}\mid}^{\Omega}_C B \iff \big|(A \cap B) \setminus C\big| \leq 1$ in Example 2.13.

Notes

This section was derived from a small part of [BY03a] by localising and simplifying it. (Note that there is no array-dividing here.) Example 2.13 is new, Example 2.14 is from Saharon Shelah. The definitions of transitivity and normality of Ω seem to be new. They were found by the author when he looked for a condition that makes Lemma 2.26 true and holds for both Ξ and Ξ_{M}. (Ξ_{M} is defined below in Definition 2.35.)

It should be noted that both Ξ and Ξ_{M} are of the form $\Omega(\Psi) = \{(\varphi, \psi) \in \Xi \mid \psi \in \Psi\}$. Yet it seems to be necessary to localise dividing in φ as well as in ψ in order to get a good theory of local rank.

2.3 Dividing patterns

For a tuple $\bar{x}$ of variables we write $\Xi{\restriction}\bar{x}$ for the set of inconsistency pairs $(\varphi, \psi) \in \Xi$ such that φ has the form $\varphi(\bar{x}; \bar{y})$, with arbitrary $\bar{y}$.

Definition 2.20. *Let $p(\bar{x})$ be a partial type over C and I a linearly ordered set.*

An I-sequence $\xi = ((\varphi^i, \psi^i))_{i \in I} \in \Xi^I$ is a dividing pattern for $p(\bar{x})$ (over C) *if there is an I-sequence $(\bar{b}^i)_{i \in I}$ that* realises ξ over C, *i. e.:*

- *$p(\bar{x}) \cup \{\varphi^i(\bar{x}; \bar{b}^i) \mid i \in I\}$ (makes sense and) is consistent, and*
- *each formula $\varphi^i(\bar{x}; \bar{b}^i)$ (φ^i, ψ^i)-divides over $C\bar{b}^{<i}$.*

If $\Delta \subseteq \Xi \restriction \bar{x}$ and $\xi \in \Delta^I$ we may call ξ a Δ-dividing pattern.

Vaguely speaking, dividing patterns measure how many dividing extensions a type has. Under certain conditions an extension of a type that admits exactly the same dividing patterns will be shown not to divide.

If I is a linearly ordered set and $i \in I$ we will temporarily write $< i$ and $\leq i$ for the initial sequences $\{j \in I \mid j < i\}$ and $\{j \in I \mid j \leq i\}$, respectively.

Theorem 2.21. *Let $p(\bar{x})$ be a partial type, definable over a set C. An I-sequence $\xi = \big(\big(\varphi^i(\bar{x}; \bar{y}^i), \psi^i(\bar{y}^i_{<k_i})\big)\big)_{i \in I} \in \Xi^I$ is a dividing pattern for $p(\bar{x})$ over C iff the following type* $\operatorname{divpat}^\xi_p\big((\bar{x}_\alpha)_{\alpha \in \omega^I}, (\bar{y}_\alpha)_{\alpha \in \omega^{\leq i}, i \in I}\big)$ *is consistent:*

$$\begin{aligned} \bigcup_{\alpha \in \omega^I} p(\bar{x}_\alpha) \quad &\cup \quad \big\{\varphi^i(\bar{x}_\alpha; \bar{y}_{\alpha \restriction \leq i}) \bigm| i \in I, \alpha \in \omega^I\big\} \\ &\cup \quad \big\{\ \psi^i(\bar{y}_{\alpha_0}, \ldots, \bar{y}_{\alpha_{k_i - 1}}) \quad \big| \quad i \in I,\ \alpha_0, \ldots, \alpha_{k_i-1} \in \omega^{\leq i}, \\ &\qquad (\alpha_0 \restriction < i) = \cdots = (\alpha_{k_i-1} \restriction < i),\ \text{and}\ \alpha_0(i) < \cdots < \alpha_{k_i-1}(i)\ \big\}. \end{aligned}$$

Before proving this theorem let us try to understand what it says. Without understanding the structure of the type $\operatorname{divpat}^\xi_p$ it is at least easy to see that it does not mention the set C. That's why the qualification 'over C' is in parentheses in Definition 2.20. The next easy observation is that the surrounding theory is not involved in the definition of $\operatorname{divpat}^\xi_p$. Hence if p and ξ make sense in a reduct T' of T, then ξ is a dividing pattern for p in the context of T iff it is one in the context of T'. We will use this to prove that simplicity and rosiness are preserved in reducts.

For understanding the structure of $\operatorname{divpat}^\xi_p$ it is perhaps best to imagine this type partially realised by tuples $(\bar{b}_\alpha)_{\alpha \in \omega^{\leq i}, i \in I}$. These tuples form a non-standard tree, and the last part of the conjunction requires that the tuples $\bar{b}_\alpha$ of level i (i.e.: $\alpha \in \omega^{\leq i}$) that define the same non-standard path $\alpha \restriction < i$ through the tree are related by the inconsistency witness ψ^i. The type $\operatorname{divpat}^\xi_p\big((\bar{x}_\alpha)_{\alpha \in \omega^I}, (\bar{b}_\alpha)_{\alpha \in \omega^{\leq i}, i \in I}\big)$ then merely expresses that for every branch $\alpha \in \omega^I$ of this tree the set $\big\{\varphi^i(\bar{x}, \bar{b}_{\alpha \restriction \leq i}) \bigm| i \in I\big\}$ is consistent with $p(\bar{x})$.

With this tree structure in mind it is easy to see that, by compactness, the property of being a dividing pattern has finite character: divpat_p^ξ is consistent iff $\mathrm{divpat}_p^{\xi\restriction J}$ is consistent for every finite $J \subseteq I$.

The tree structure of divpat_p^ξ already suggests a proof strategy.

Proof. We will prove the equivalence of the following statements:

(1) ξ is a dividing pattern for p over C.

(2) divpat_p^ξ is consistent.

(3) The type

$$\begin{aligned}
&\mathrm{divpat}'^{\xi}_{p}\big((\bar{x}_\alpha)_{\alpha\in\omega^I}, (\bar{y}_\alpha)_{\alpha\in\omega^{\leq i}, i\in I}\big)\\
&\quad = \mathrm{divpat}_p^\xi\left((\bar{x}_\alpha)_{\alpha\in\omega^I}, (\bar{y}_\alpha)_{\alpha\in\omega^{\leq i}, i\in I}\right)\\
&\quad \cup \left\{\bar{y}_\alpha \equiv_{C\{\bar{y}_{\alpha\restriction\leq j} \mid j<i\}} \bar{y}_{\alpha'} \;\middle|\; i\in I,\ \alpha,\alpha'\in\omega^{\leq i},\ (\alpha\restriction<i) = (\alpha'\restriction<i)\right\}
\end{aligned}$$

is consistent.

We first prove that (3) implies (1): Let the tuples $(\bar{b}_\alpha)_{\alpha\in\omega^{\leq i}, i\in I}$ be a partial realisation of $\mathrm{divpat}'^{\xi}_{p}$. For $i \in I$ write ζ^i for the unique function $\zeta^i \in \{0\}^{<i}$, and for $m < \omega$ write $\zeta^i{}^\frown(m)$ for the extension of ζ^i that maps i to m. Then for every $i \in I$ the sequence $(\bar{b}_{\zeta^i{}^\frown(m)})_{m<\omega}$ witnesses that $\bar{b}_{\zeta^i{}^\frown(0)}$ (φ^i, ψ^i)-divides over $C\{\bar{b}_{\zeta^j{}^\frown(0)} \mid j < i\}$. Hence the I-sequence $(\bar{b}_{\zeta^i{}^\frown(0)})_{i\in I}$ realises ξ over C.

Next we observe that we need only prove that (1) implies (2) and that (2) implies (3) in case I is finite. The general case then follows by compactness. Thus we can use induction on the size of I.

The case $I = \emptyset$ is trivial: $() \in \Xi^0$ is a dividing pattern for p over C iff p is consistent, and we have $\mathrm{divpat}_p^{()} = \mathrm{divpat}'^{()}_{p} = p(\bar{x}_{()})$.

Now suppose the implications (1) $\implies$ (2) $\implies$ (3) hold for I, and we are given $((\varphi^s(\bar{x};\bar{y}), \psi^s(\bar{y}_{<k})))^\frown\xi \in \Xi^{\{s\}\cup I}$, where $s \notin I$ is less than every element of I. It is not hard to see that (1) $\implies$ (2) $\implies$ (3) for $((\varphi^s, \psi^s))^\frown\xi$, using the following three easy facts:

(i) $((\varphi^s, \psi^s))^\frown\xi$ is a dividing pattern for p over C iff there is a tuple $\bar{b}$ such that $\varphi^s(\bar{x};\bar{b})$ (φ^s, ψ^s)-divides over C, and ξ is a dividing pattern for $p(\bar{x}) \cup \varphi^s(\bar{x};\bar{b})$.

(ii) $\mathrm{divpat}_p^{((\varphi^s,\psi^s))^\frown\xi}$ is consistent iff there is a sequence $(\bar{b}_m)_{m<\omega}$ such that $\models \psi^s(\bar{b}_{m_0}, \dots, \bar{b}_{m_{k-1}})$ for any $m_0 < \cdots < m_{k-1} < \omega$ and the type $\mathrm{divpat}^\xi_{p(\bar{x})\cup\varphi^s(\bar{x};\bar{b}_m)}$ is consistent for every $m < \omega$.

(iii) $\mathrm{divpat}'^{((\varphi^s,\psi^s))^\frown\xi}_{p}$ is consistent iff there is a sequence $(\bar{b}_m)_{m<\omega}$ such that $\models \psi^s(\bar{b}_{m_0}, \dots, \bar{b}_{m_{k-1}})$ for any $m_0 < \cdots < m_{k-1} < \omega$, $\bar{b}_m \equiv_C \bar{b}_0$ for all

$m < \omega$, and the type $\mathrm{divpat}'^{\xi}_{p(\bar{x}) \cup \varphi^s(\bar{x};\bar{b}_m)}$ is consistent for every $m < \omega$. (Thus the sequence $(\bar{b}_m)_{m<\omega}$ witnesses that $\bar{b}_0$ (φ^s, ψ^s)-divides over C.) $\square$

Exercises

Exercise 2.22. (tree property)

A formula $\varphi(\bar{x};\bar{y})$ *has the tree-property (of order k)* if there is a tree of tuples $(\bar{b}_\alpha)_{\alpha\in\omega^{<\omega}}$ such that for every limit point $\alpha \in \omega^\omega$ the branch $\{\varphi(\bar{x};\bar{b}_{\alpha\restriction n}) \mid n < \omega\}$, is consistent, and at every node $\alpha \in \omega^{<\omega}$ the set of successors $\{\varphi(\bar{x};\bar{b}_{\alpha^\frown(i)}) \mid i < \omega\}$ is k-inconsistent (i. e., every subset with k elements is inconsistent).

Show that formula $\varphi(\bar{x};\bar{y})$ has the tree-property of order k if and only if there is a k-inconsistency witness $\psi(\bar{y}_{<k})$ for φ such that $\mathrm{D}_{\varphi,\psi}(\emptyset) = \infty$. Here $\mathrm{D}_{\varphi,\psi}$ is D_Δ as defined in Definition 2.23 below for the case $\Delta = \{(\varphi,\psi)\}$.

Notes

The only new things in this section are the *term* ‘dividing pattern’ and the idea of admitting arbitrary linear orders in order to allow a uniform treatment of dividing patterns and the tree property.

Dividing patterns appear in [BY03b] in the following guise: Let α be an ordinal and $I = \alpha^{\mathrm{opp}}$, i. e., α with the opposite order. Then $\xi \in \Xi^I$ is a dividing pattern for p iff $\xi \in \mathrm{D}(p,\Xi)$ in the notation of [BY03b, Definition 1.8]. The idea that ξ being a dividing pattern can be expressed by a partial type is also from [BY03b].

A realisation of a dividing pattern $\xi \in \Xi^\alpha$ is also the same thing as a dividing chain as defined in [Cas99].

2.4 Local rank and symmetry

Suppose $\Delta \subseteq \Xi\restriction\bar{x}$ is finite. If there are arbitrarily long finite Δ-dividing patterns for p, then there is an inconsistency pair $(\varphi,\psi) \in \Delta$ such that there are arbitrarily long finite (φ,ψ)-dividing patterns for p. It follows that $(\varphi,\psi)^I$ is a Δ-dividing pattern for p for every linearly ordered set I. Therefore the following definition makes sense:

Definition 2.23. *Let $p(\bar{x})$ be a partial type and $\Delta \subseteq \Xi\restriction\bar{x}$ a finite set of inconsistency pairs $(\varphi(\bar{x};\bar{y}_\varphi),\psi)$. Then $\mathrm{D}_\Delta(p) \in \omega \cup \{\infty\}$ is ∞ if p has Δ-dividing patterns of arbitrary order type, or otherwise the greatest number $n < \omega$ such that Δ-dividing patterns of length n exist for p.*

If $p = \mathrm{tp}(\bar{a}/B)$ we abbreviate $\mathrm{D}_\Delta(\bar{a}/B) = \mathrm{D}_\Delta(p)$.

Remark 2.24. *For any $\bar{a}$, B, C and finite $\Delta \subseteq \Xi{\restriction}\bar{x}$:*

$$\mathrm{D}_\Delta(\bar{a}/BC) \leq \mathrm{D}_\Delta(\bar{a}/C).$$

Proof. Let $p = \mathrm{tp}(\bar{a}/BC)$. If $\xi \in \Delta^n$ is a dividing pattern for p, then divpat_p^ξ is consistent. Hence $\mathrm{divpat}_{p\restriction C}^\xi$ is consistent, so ξ is a dividing pattern for $p{\restriction}C$. $\square$

For the rest of this section we fix a set $\Omega \subset \Xi$ that is closed under variable substitution.

Lemma 2.25. *Suppose $\mathrm{D}_\Delta(\bar{a}/BC) = \mathrm{D}_\Delta(\bar{a}/C) < \infty$ for all finite $\Delta \subseteq \Omega{\restriction}\bar{x}$. Then $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{\Omega*}_C B$.*

Proof. Towards a contradiction, suppose $\bar{a} \not\mathop{\smile\!\!\!\!\!\!|}^{\Omega*}_C B$. Then there is a set $\hat{B} \supset B$ such that $\bar{a}' \not\mathop{\smile\!\!\!\!\!\!|}^{\Omega}_C \hat{B}$ holds for every $\bar{a}'$ realising $p(\bar{x}) = \mathrm{tp}(\bar{a}/BC)$. Hence the set

$$p(\bar{x}) \cup \left\{ \neg\varphi(\bar{x};\bar{b}) \;\middle|\; (\varphi,\psi) \in \Delta, \text{ and } \varphi(\bar{x};\bar{b}) \ \ (\varphi,\psi)\text{-divides over } C \right\}$$

is inconsistent. By compactness there are inconsistency pairs $(\varphi^i,\psi^i) \in \Delta$ and tuples $\bar{b}^i$ such that $p(\bar{x}) \vdash \bigvee_{i<k} \varphi^i(\bar{x};\bar{b}^i)$ and $\varphi^i(\bar{x};\bar{b}^i)$ (φ^i,ψ^i)-divides over C.

Let ξ be a Δ-dividing pattern for p of maximal length $|\xi| = \mathrm{D}_\Delta(p)$, realised over $C\bar{b}^0\bar{b}^1\ldots\bar{b}^{k-1}$ by, say, $(\bar{b}_j)_{j<|\xi|}$. Let $\bar{a}'$ realise $p(\bar{x}) \cup \{\varphi_j(\bar{x};\bar{b}_j) \mid j < |\xi|\}$. Then $\models \varphi^i(\bar{a}';\bar{b}^i)$ for an index $i < k$. Hence $(\varphi^i,\psi^i)^\frown\xi$ is a Δ-dividing pattern for p, realised by $\bar{b}^{i\frown}(\bar{b}_j)_{j<|\xi|}$. This contradicts maximality of $|\xi|$. $\square$

Having shown a connection between $\mathop{\smile\!\!\!\!\!\!|}^{\Omega}$ and the D_Δ-ranks under a combinatorial condition, we now show a sort of converse under a geometric condition.

Lemma 2.26. *Suppose Ω is transitive and normal and $B \mathop{\smile\!\!\!\!\!\!|}^{\Omega*}_C \bar{a}$. Then for every finite $\Delta \subseteq \Omega{\restriction}\bar{x}$ we have $\mathrm{D}_\Delta(\bar{a}/BC) = \mathrm{D}_\Delta(\bar{a}/C)$.*

Proof. Since $\mathrm{D}_\Delta(\bar{a}/BC) \leq \mathrm{D}_\Delta(\bar{a}/C)$ holds anyway we need only prove that $\mathrm{D}_\Delta(\bar{a}/C) \geq n$ implies $\mathrm{D}_\Delta(\bar{a}/BC) \geq n$. By definition of D_Δ there is a Δ-dividing pattern $\xi = ((\varphi^i,\psi^i))_{i<n} \in \Delta^n$ for $\mathrm{tp}(\bar{a}/C)$, and this is witnessed by tuples $(\bar{b}^i)_{i<n}$ such that $\models \varphi^i(\bar{a};\bar{b}^i)$ and $\varphi^i(\bar{x};\bar{b}^i)$ (φ^i,ψ^i)-divides over $C\bar{b}^{<i}$ for all $i < n$. Since $B \mathop{\smile\!\!\!\!\!\!|}^{\Omega*}_C \bar{a}$ we may assume that $B \mathop{\smile\!\!\!\!\!\!|}^{\Omega}_C \bar{a}\bar{b}^{<n}$. Hence $BC\bar{b}^{<i} \mathop{\smile\!\!\!\!\!\!|}^{\Omega}_{C\bar{b}^{<i}} \bar{b}_i$ by base monotonicity and normality for all $i < n$. Now since $\varphi^i(\bar{x};\bar{b}^i)$ (φ^i,ψ^i)-divides over $C\bar{b}^{<i}$ we get by transitivity that $\varphi^i(\bar{x};\bar{b}^i)$ (φ^i,ψ^i)-divides over $BC\bar{b}^{<i}$ as well. Therefore $\bar{b}^{<n}$ also witnesses that ξ is a dividing pattern for $\mathrm{tp}(\bar{a}/BC)$, so $\mathrm{D}(\bar{a}/BC) \geq n$. $\square$

Theorem 2.27. *Suppose Ω is transitive and normal, and $\mathrm{D}_\Delta(\emptyset) < \infty$ for all finite $\Delta \subseteq \Omega{\restriction}\bar{x}$. Then the following conditions are equivalent:*

(1) $\bar{a} \mathop{\smile\!\!\!\!\!\!\;\mid}^{\Omega*}_C B$.

(2) $\mathrm{D}_\Delta(\bar{a}/BC) = \mathrm{D}_\Delta(\bar{a}/C)$ *for all finite* $\Delta \subseteq \Omega{\restriction}\bar{x}$.

(3) $B \mathop{\smile\!\!\!\!\!\!\;\mid}^{\Omega*}_C \bar{a}$.

Proof. (3) implies (2) by Lemma 2.26, and (2) implies (1) by Lemma 2.25. Hence $\mathop{\smile\!\!\!\!\!\!\;\mid}^{\Omega*}$ is symmetric, so (1) implies (3). □

Notes

Theorem 2.27 was proved by Byunghan Kim in the case $\Omega = \Xi$, cf. [Kim96, Theorem 5.1], and by Alf Onshuus in the case $\Omega = \Xi_{\mathrm{M}}$, cf. [Ons03a, Theorem 4.1.3]. (To be pedantic, both Kim and Onshuus prove their results for ranks that are very similar to, but not exactly the same as, our ranks D_Δ.)

2.5 A better theorem on local forking

Like in the previous section we fix a set $\Omega \subseteq \Xi$ which is closed under variable substitution.

Lemma 2.28. *The following statements are equivalent:*

(1) $\mathop{\smile\!\!\!\!\!\!\;\mid}^{\Omega*}$ *satisfies the local character axiom.*

(2) $\mathop{\smile\!\!\!\!\!\!\;\mid}^{\Omega}$ *satisfies the local character axiom.*

(3) $\mathrm{D}_{\varphi,\psi}(\emptyset) < \infty$ *for every* $(\varphi, \psi) \in \Omega$.

Proof. (1) implies (2): This follows from $A \mathop{\smile\!\!\!\!\!\!\;\mid}^{\Omega*}_C B \implies A \mathop{\smile\!\!\!\!\!\!\;\mid}^{\Omega}_C B$.

(2) implies (3): Suppose $\mathop{\smile\!\!\!\!\!\!\;\mid}^{\Omega}$ has local character with a constant κ, but $\mathrm{D}_{\varphi,\psi}(\emptyset) = \infty$. We may assume that κ is regular. $(\varphi, \psi)^\kappa$ is a dividing pattern for the empty type, so it has a realisation $(\bar{b}_i)_{i<\kappa}$ over $\emptyset$. Let $\bar{a}$ realise $\{\varphi(\bar{x}; \bar{b}_i) \mid i < \kappa\}$. By local character there is a subset $C \subseteq \bar{b}_{<\kappa}$ such that $\bar{a} \mathop{\smile\!\!\!\!\!\!\;\mid}^{\Omega}_C \bar{b}_{<\kappa}$ and $|C| < \kappa$. Since κ is regular, there is $\alpha < \kappa$ such that $C \subseteq \bar{b}_{<\alpha}$. Hence $\bar{a} \mathop{\smile\!\!\!\!\!\!\;\mid}^{\Omega}_{\bar{b}_{<\alpha}} \bar{b}_{<\kappa}$, a contradiction to the fact that $\models \varphi(\bar{a}; \bar{b}_\alpha)$ holds and $\varphi(\bar{x}; \bar{b}_\alpha)$ (φ, ψ)-divides over $\bar{b}_{<\alpha}$.

(3) implies (1): Note that $\mathrm{D}_\Delta(p(\bar{x})) < \omega$ for all finite $\Delta \subseteq \Omega{\restriction}\bar{x}$ and partial types $p(\bar{x})$. We will prove local character for $\mathop{\smile\!\!\!\!\!\!\;\mid}^{\Omega*}$ with $\kappa = |T|^+$. So suppose we have a type $p(\bar{x}) = \mathrm{tp}(\bar{a}/B)$ with finite $\bar{a}$.

For every finite $\Delta \subseteq \Omega\restriction\bar{x}$ we can find a finite subset $C_\Delta \subseteq B$ such that $\mathrm{D}_\Delta(p\restriction C_\Delta) = \mathrm{D}_\Delta(p)$: For each Δ-dividing pattern ξ of length $|\xi| = \mathrm{D}_\Delta(p)+1$ (there are only finitely many) the type divpat_p^ξ is inconsistent, so there is a finite subset $C_\xi \subseteq B$ such that $\mathrm{divpat}_{p\restriction C_\xi}^\xi$ is still inconsistent. If C_Δ is the union of these sets C_ξ, then clearly C_Δ is a finite set such that $\mathrm{D}_\Delta(p\restriction C_\Delta) = \mathrm{D}_\Delta(p)$.

Now let C be the union of these sets C_Δ for all finite $\Delta \subseteq \Omega\restriction\bar{x}$. Then $|C| \leq |T|$, so $|C| < \kappa$. Moreover, $\mathrm{D}_\Delta(p\restriction C) = \mathrm{D}_\Delta(p\restriction C_\Delta) = \mathrm{D}_\Delta(p)$ for all finite $\Delta \subseteq \Omega\restriction\bar{x}$. Hence $\bar{a} \mathop{\smile\hspace{-0.9em}|\hspace{0.6em}}^{\Omega*}_C B$ by Lemma 2.25. $\square$

Now we are prepared for the following improved version of Theorem 1.18 for the case $\mathop{\smile\hspace{-0.9em}|\hspace{0.6em}} = \mathop{\smile\hspace{-0.9em}|\hspace{0.6em}}^{\Omega}$:

Theorem 2.29. *Suppose $\mathop{\smile\hspace{-0.9em}|\hspace{0.6em}}^{\Omega}$ satisfies the transitivity and normality axioms. Then $\mathop{\smile\hspace{-0.9em}|\hspace{0.6em}}^{\Omega*}$ is an independence relation if and only if the following, equivalent, conditions are satisfied:*

- **(1)** *$\mathop{\smile\hspace{-0.9em}|\hspace{0.6em}}^{\Omega*}$ satisfies the local character axiom.*
- **(2)** *$\mathop{\smile\hspace{-0.9em}|\hspace{0.6em}}^{\Omega}$ satisfies the local character axiom.*
- **(3)** $\mathrm{D}_{\varphi,\psi}(\emptyset) < \infty$ *for every* $(\varphi,\psi) \in \Omega$.
- **(4)** $A \mathop{\smile\hspace{-0.9em}|\hspace{0.6em}}^{\Omega*}_C B$ *implies* $B \mathop{\smile\hspace{-0.9em}|\hspace{0.6em}}^{\Omega*}_C A$.
- **(5)** $A \mathop{\smile\hspace{-0.9em}|\hspace{0.6em}}^{\Omega*}_C B$ *implies* $B \mathop{\smile\hspace{-0.9em}|\hspace{0.6em}}^{\Omega}_C A$.

Proof. By Proposition 2.12 (and since we have assumed transitivity and normality), $\mathop{\smile\hspace{-0.9em}|\hspace{0.6em}}^{\Omega}$ satisfies the first five axioms for independence relations and strong finite character. Hence by Lemmas 1.17 and 2.2, the relation $\mathop{\smile\hspace{-0.9em}|\hspace{0.6em}}^{\Omega*}$ satisfies all axioms except local character. Therefore it is an independence relation if and only if (1) holds.

Conditions (1) to (3) are equivalent by Lemma 2.28. If $\mathop{\smile\hspace{-0.9em}|\hspace{0.6em}}^{\Omega*}$ is an independence relation, then (4) holds by Theorem 2.27. (4) implies (5) because $B \mathop{\smile\hspace{-0.9em}|\hspace{0.6em}}^{\Omega*}_C A$ implies $B \mathop{\smile\hspace{-0.9em}|\hspace{0.6em}}^{\Omega}_C A$.

(5) implies (2): We choose $\kappa(A) = (|T| + |A|)^+$ for every set A. Given sets A and B, let $\bar{b}$ be an enumeration of B. By Remark 2.4 there is a subset $C \subseteq B$ such that $|C| < \kappa(A)$ and $\mathrm{tp}(\bar{b}/AC)$ is finitely realised in C. Hence by Remark 2.3 we have $B \mathop{\smile\hspace{-0.9em}|\hspace{0.6em}}^{\mathrm{f}}_C A$, so $A \mathop{\smile\hspace{-0.9em}|\hspace{0.6em}}^{\mathrm{d}}_C B$ by (5). $\square$

Note that if Ω satisfies the slightly stronger conditions of transitivity and normality (and it will do so in our applications), then Theorem 2.27 gives us additional information on $\mathop{\smile\hspace{-0.9em}|\hspace{0.6em}}^{\Omega*}$:

Exercises

Exercise 2.30. (symmetry of $\mathop{\smile\hspace{-0.9em}\mid}^{\Omega}$)

Suppose $\mathop{\smile\hspace{-0.9em}\mid}^{\Omega}$ satisfies transitivity, normality and symmetry. Then $\mathop{\smile\hspace{-0.9em}\mid}^{\Omega}$ is an independence relation, and $\mathop{\smile\hspace{-0.9em}\mid}^{\Omega} = \mathop{\smile\hspace{-0.9em}\mid}^{\Omega *}$.

Notes

This section extends some standard results to a more general context.

2.6 A better theorem on Shelah-forking

In order to avoid the use of exercises in the main text (Exercises 1.24 and 2.18 in this case), we give a direct proof of the following result:

Proposition 2.31. $\mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{d}} = \mathop{\smile\hspace{-0.9em}\mid}^{\Xi}$.

Proof. First suppose $A \mathop{\not\smile\hspace{-0.9em}\mid}^{\Xi}_{C} B$, so there is $\bar{a} \in A$ and $\big(\varphi(\bar{x};\bar{y}), \psi(\bar{y}_{<\omega})\big) \in \Xi$ such that $\mathrm{tp}(\bar{a}/BC)$ (φ,ψ)-divides over C. Hence there is $\bar{b} \in BC$ such that $\varphi(\bar{x};\bar{b}) \in \mathrm{tp}(\bar{a}/BC)$ and $\varphi(\bar{x};\bar{b})$ (φ,ψ)-divides over C. Let $\bar{b}_{<\omega}$ be a sequence witnessing this, i.e., each $\bar{b}_i$ realises $\mathrm{tp}(\bar{b}/C)$ and $\models \psi(\bar{b}_{i_0},\dots \bar{b}_{i_{k-1}})$ holds for all $i_0 < \ldots < i_{k-1} < \omega$. We can extend this sequence and extract a sequence of C-indiscernibles from it, so we may assume that $\bar{b}_{<\omega}$ is C-indiscernible. Moreover, we may assume that $\bar{b}_0 = \bar{b}$.

Towards a contradiction, suppose $A \mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{d}}_{C} B$. Then there would be $\bar{a}' \equiv_{BC} \bar{a}$ such that the sequence is $\bar{a}'C$-indiscernible. But then $\models \big(\bigwedge_{i<k} \varphi(\bar{a}';\bar{b}_i)\big) \wedge \psi(\bar{b}_{<k})$, contradicting the fact that (φ,ψ) is a k-inconsistency witness.

For the converse, suppose $A \mathop{\not\smile\hspace{-0.9em}\mid}^{\mathrm{d}}_{C} B$, so there is a sequence $\bar{b}_{<\omega}$ of C-indiscernibles with $\bar{b}_0 \in BC$ that cannot be $A'C$-indiscernible for any $A' \equiv_{BC} A$. By compactness this must be due to a formula $\varphi(\bar{a};\bar{b}_0)$ ($\bar{a} \in A$) and a formula $\psi(\bar{b}_{<k})$ such that $\big(\bigwedge_{i<k} \varphi(\bar{x};\bar{b}_0)\big) \wedge \psi(\bar{b}_{<k})$ is inconsistent. So $\big(\varphi(\bar{x};\bar{y}), \psi(\bar{y}_{<k})\big)$ is a k-inconsistency witness and $\mathrm{tp}(\bar{a}/BC)$ (φ,ψ)-divides over C. Hence $A \mathop{\not\smile\hspace{-0.9em}\mid}^{\Xi}_{C} B$ as well. $\square$

Theorem 2.32. *A complete consistent theory T is simple if and only if the following, equivalent, conditions are satisfied:*

(1) $\mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{f}}$ *satisfies the local character axiom.*

(2) $\mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{d}}$ *satisfies the local character axiom.*

(3) $\mathrm{D}_{\varphi,\psi}(\emptyset) < \omega$ *for each* $(\varphi,\psi) \in \Xi$.

(4) $\mathop{\smile\!\!\!\!\!|}^{\mathrm{f}}$ *is symmetric.*

(5) $A \mathop{\smile\!\!\!\!\!|}^{\mathrm{f}}_C B$ *implies* $B \mathop{\smile\!\!\!\!\!|}^{\mathrm{d}}_C A$.

(6) $\mathop{\smile\!\!\!\!\!|}^{\mathrm{d}}$ *is symmetric.*

Moreover, in a simple theory $\mathop{\smile\!\!\!\!\!|}^{\mathrm{f}} = \mathop{\smile\!\!\!\!\!|}^{\mathrm{d}}$ *is the finest independence relation.*

Proof. Simplicity is equivalent to (1) by Theorem 1.23. For the equivalence of (1)–(5) note that $\mathop{\smile\!\!\!\!\!|}^{\mathrm{f}} = \mathop{\smile\!\!\!\!\!|}^{\mathrm{d}*} = \mathop{\smile\!\!\!\!\!|}^{\Xi*}$. Then apply Theorem 2.29. (6) clearly implies (5). The converse holds since (6) implies $\mathop{\smile\!\!\!\!\!|}^{\mathrm{f}} = \mathop{\smile\!\!\!\!\!|}^{\mathrm{d}}$ by Theorem 1.23. The 'moreover' statement is also from Theorem 1.23. □

Corollary 2.33. *Every reduct of a simple theory is simple.*
T is simple if and only if T^{eq} is simple.

Proof. We first show that a reduct of a simple theory is simple. We already know that a theory T is simple iff $\mathrm{D}_{\varphi,\psi}(\emptyset) < \infty$ holds for every inconsistency pair (φ, ψ). Let T' be a reduct of T. For formulas φ and ψ in the signature of T', (φ, ψ) is an inconsistency pair for T' if and only if it is an inconsistency pair for T. Moreover, $\mathrm{D}_{\varphi,\psi}(\emptyset)$ is the maximal $n < \omega$ such that for the unique $\xi \in \{(\varphi, \psi)\}^n$ the type divpat_p^ξ from Theorem 2.21 is consistent. Since this type is independent of the ambient theory, it does not matter whether we evaluate $\mathrm{D}_{\varphi,\psi}$ in T or in T'. Thus if T is simple then so is T'.

One consequence is that if T^{eq} is simple, then so is its reduct T. We now show the converse. So suppose T is simples and (φ, ψ) is a k-inconsistency pair for T^{eq}. We may assume that as much as possible is coded in a single imaginary variable, so $\varphi \equiv \varphi(x; y)$ and $\psi \equiv \psi(y_{<k})$. The sorts of x and y correspond to definable equivalence relations ϵ_x and ϵ_y. Now consider $\varphi'(\bar{x}; \bar{y}) \equiv \varphi(\bar{x}/\epsilon_x; \bar{y}/\epsilon_y)$ and $\psi'(\bar{y}_{<k}) \equiv \psi(\bar{y}_0/\epsilon_y, \ldots, \bar{y}_{k-1}/\epsilon_y)$. φ' and ψ' can be expressed in T, and (φ', ψ') is a k-inconsistency pair for T. Clearly $\mathrm{D}_{\varphi,\psi}(\emptyset) = \mathrm{D}_{\varphi',\psi'}(\emptyset)$, so T^{eq} also satisfies condition (3) of Theorem 2.32. □

The following remark shows that we can also apply Theorem 2.27 to get an alternative characterisation of $\mathop{\smile\!\!\!\!\!|}^{\mathrm{f}}$ in a simple theory.

Remark 2.34. *Ξ is transitive and normal.*

Proof. For transitivity of Ξ suppose $C \mathop{\smile\!\!\!\!\!|}^{\Xi}_D \bar{a}_0$, $D \subseteq C$ and $\varphi(\bar{y}; \bar{a}_0)$ (φ, ψ)-divides over D, witnessed by $\bar{a}_{<\omega}$. By compactness and Fact 1.11 we may assume that $\bar{a}_{<\omega}$ is D-indiscernible. Since $C \mathop{\smile\!\!\!\!\!|}^{\mathrm{d}}_D \bar{a}_0$ we may assume that $\bar{a}_{<\omega}$ is in fact C-indiscernible. Thus $\bar{a}_{<\omega}$ witnesses that $\varphi(\bar{y}; \bar{a}_0)$ (φ, ψ)-divides over C.

For normality of Ξ just observe that $\psi(\bar{y}_{<k})$ is a k-inconsistency witness for $\varphi(\bar{x}, \bar{z}; \bar{y})$ iff $\psi(\bar{y}_{<k}) \wedge z_0 = z_1 = \ldots z_{k-1}$ is a k-inconsistency witness for $\varphi(\bar{x}; \bar{z}, \bar{y})$. □

Notes

Apart from the style of presentation, nothing is new in this section.

2.7 A better theorem on thorn-forking

Definition 2.35. *We define the following subset of Ξ:*

$$\Xi_{\mathrm{M}} = \Big\{ \big(\varphi(\bar{x}; u\bar{v}), \psi((u\bar{v})_{<k})\big) \in \Xi \;\Big|\; \psi((u\bar{v})_{<k}) \quad \equiv \quad \bigwedge\nolimits_{i<j<k}(u_i \neq u_j \wedge \bar{v}_i = \bar{v}_j) \Big\}.$$

Note that if $\psi((u\bar{v})_{<k})$ (as in the definition) is a k-inconsistency witness for $\varphi(\bar{x}; u\bar{v})$, then whenever $\varphi(\bar{a}; g\bar{h})$ holds, g must be algebraic over $\bar{a}\bar{h}$.

Proposition 2.36. *Some properties of $\mathop{\smile\!\!\!\!\!\!|}^{\Xi_{\mathrm{M}}}$:*

(1) $\mathop{\smile\!\!\!\!\!\!|}^{\Xi_{\mathrm{M}}}$ *has the following characterisation:*

$$A \mathop{\smile\!\!\!\!\!\!|}^{\Xi_{\mathrm{M}}}_{C} B \iff \left(\begin{array}{l} \operatorname{acl}(AD) \cap B \subseteq \operatorname{acl} D \\ \qquad \textit{for every set } D \textit{ such that } C \subseteq D \subseteq BC \end{array} \right).$$

(2) Ξ_{M} *is transitive and normal.*

(3) $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}}_{C} B$ *implies* $A \mathop{\smile\!\!\!\!\!\!|}^{\Xi_{\mathrm{M}}}_{C} B$.

(4) $\mathop{\smile\!\!\!\!\!\!|}^{\text{þ}} = \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}*} = \mathop{\smile\!\!\!\!\!\!|}^{\Xi_{\mathrm{M}}*}$.

Proof. (1) Suppose there is a set D such that $C \subseteq D \subseteq BC$ and $\operatorname{acl}(\bar{a}D) \cap B \not\subseteq \operatorname{acl} D$. So there is an element $e \in \operatorname{acl}(\bar{a}D) \cap B \setminus \operatorname{acl} D$. Let $\alpha(u, \bar{a}, \bar{d})$ with $\bar{d} \in D$ be an algebraic formula realised by e.

Then for some $k < \omega$, $\models \varphi(\bar{a}; e\bar{d})$ holds, where $\varphi(\bar{x}; u\bar{v}) \equiv \alpha(u, \bar{x}, \bar{v}) \wedge \exists_{<k} u' \alpha(u', \bar{x}, \bar{v})$. We set $\psi((u\bar{v})_{<k}) \equiv \bigwedge_{i<j<k}(u_i \neq u_j \wedge \bar{v}_i = \bar{v}_j)$. Clearly $(\varphi, \psi) \in \Xi_{\mathrm{M}}$.

Let $e_{<\omega}$ be a sequence of distinct realisations of the (non-algebraic) type $\operatorname{tp}(e/D)$. Then the sequence $(e_i\bar{d})_{i<\omega}$ witnesses that $\operatorname{tp}(\bar{a}/BC)$ (φ, ψ)-divides over C. Hence $\bar{a} \not\mathop{\smile\!\!\!\!\!\!|}^{\Xi_{\mathrm{M}}}_{C} B$.

Conversely, suppose $\bar{a} \not\mathop{\smile\!\!\!\!\!\!|}^{\Xi_{\mathrm{M}}}_{C} B$. So $\operatorname{tp}(\bar{a}/BC)$ (φ, ψ)-divides over C for some $(\varphi, \psi) \in \Xi_{\mathrm{M}}$. Let this be witnessed by $(e_i\bar{d})_{i<\omega}$. We may assume that $e_0\bar{d} \in BC$.

Since $e_i\bar{d} \equiv_C e_j\bar{d}$ for $i < j < \omega$, $e_i \equiv_{C\bar{d}} e_j$ holds as well, so the sequence $e_{<\omega}$ witnesses that $e_0 \notin \operatorname{acl}(C\bar{d})$. In particular, $e_0 \in BC \setminus C$, so $e_0 \in B$.

Moreover, $\models \varphi(\bar{a}; e_0\bar{d})$ implies that $e_0 \in \operatorname{acl}(\bar{a}\bar{d}) \subseteq \operatorname{acl}(\bar{a}C\bar{d})$. So choosing $D = C\bar{d}$ we get $e_0 \in \operatorname{acl}(\bar{a}D) \cap B \setminus \operatorname{acl} D$.

(2) For transitivity suppose $D \subseteq C$, $C \mathop{\smile\hspace{-0.9em}\mid}^{\Xi_{\mathrm{M}}}_{D} g\bar{h}$ and $\varphi(\bar{y}; \bar{a})$ (φ, ψ)-divides over D, where $(\varphi, \psi) \in \Xi_{\mathrm{M}}$. Note that $\bar{a} = g\bar{h}$ and g is not algebraic over $D\bar{h}$. By (1) we have $\operatorname{acl}(C\bar{h}) \cap \{g\} \subseteq \operatorname{acl}(D\bar{h})$, so g is not algebraic over $C\bar{h}$. Hence there is a sequence $(\bar{g}_i)_{i<\omega}$ of distinct elements $g_i \equiv_{C\bar{h}} g$. It is easy to see that the sequence $(g_i\bar{h})_{i<\omega}$ witnesses the fact that $\varphi(\bar{y}; \bar{a})$ (φ, ψ)-divides over C.

For normality just observe that $\varphi(\bar{x}, \bar{z}; \bar{y})$ and $\psi(\bar{y}_{<k})$ are of the form necessary for $(\varphi, \psi) \in \Xi_{\mathrm{M}}$ by Definition 2.35 iff $\varphi(\bar{x}; \bar{z}, \bar{y})$ and $\psi'(\bar{y}_{<k}, \bar{z}_{<k}) \equiv \psi(\bar{y}) \wedge z_0 = z_1 = \cdots = z_{k-1}$ are of this form.

(3) Suppose $A \mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{M}}_{C} B$. So $\operatorname{acl}(AD) \cap \operatorname{acl}(BD) \subseteq \operatorname{acl} D$ for every set D such that $C \subseteq D \subseteq \operatorname{acl}(BC)$. Hence $\operatorname{acl}(AD) \cap B \subseteq \operatorname{acl} D$ for every set D such that $C \subseteq D \subseteq BC$.

(4) $\mathop{\smile\hspace{-0.9em}\mid}^{\text{þ}} = \mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{M}*}$ by definition. $A \mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{M}*}_{C} B$ implies $A \mathop{\smile\hspace{-0.9em}\mid}^{\Xi_{\mathrm{M}}*}_{C} B$ by (2).

For the converse suppose $A \mathop{\smile\hspace{-0.9em}\mid}^{\Xi_{\mathrm{M}}*}_{C} B$ holds and we are given $\hat{B} \supseteq B$. Let $A' \equiv_{BC} A$ be such that $A \mathop{\smile\hspace{-0.9em}\mid}^{\Xi_{\mathrm{M}}}_{C} \operatorname{acl}(\hat{B}C)$. Then $\operatorname{acl}(A'D) \cap \operatorname{acl}(\hat{B}C) \subseteq \operatorname{acl} D$ for every set D such that $C \subseteq D \subseteq \operatorname{acl}(\hat{B}C)$.

Since $\operatorname{acl}(\hat{B}C) = \operatorname{acl}(\hat{B}D)$, and since $\operatorname{acl} D \subseteq \operatorname{acl}(AD) \cap \operatorname{acl}(\hat{B}D)$ holds trivially, $A' \mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{M}}_{C} \operatorname{acl}(\hat{B}C)$ follows. ☐

It is not true in general that $\mathop{\smile\hspace{-0.9em}\mid}^{\Xi_{\mathrm{M}}} = \mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{M}}$: Let T be the theory of an everywhere infinite forest, as in Example 1.31. Let a, b_0 and b_1 be nodes such that $\models Rab_0$, $\models Rab_1$ and $\models b_0 \neq b_1$. Then $a \not\mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{M}}_{\emptyset} b_0b_1$ because $a \in \operatorname{acl}(a) \cap \operatorname{acl}(b_0b_1) \setminus \operatorname{acl} \emptyset$. But $a \mathop{\smile\hspace{-0.9em}\mid}^{\Xi_{\mathrm{M}}}_{\emptyset} b_0b_1$ holds. This can be verified by checking $\operatorname{acl}(aD) \cap \{b_0, b_1\} \subseteq \operatorname{acl} D$ for the four possible values of D such that $\emptyset \subseteq D \subseteq \{b_0, b_1\}$.

Theorem 2.37. *$\mathop{\smile\hspace{-0.9em}\mid}^{\text{þ}}$ is an independence relation for T if and only if the following, equivalent, conditions are satisfied:*

(1) *$\mathop{\smile\hspace{-0.9em}\mid}^{\text{þ}}$ satisfies the local character axiom.*

(2) *$\mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{M}}$ satisfies the local character axiom.*

(3) *$\mathrm{D}_{\varphi,\psi}(\emptyset) < \infty$ for every $(\varphi, \psi) \in \Xi_{\mathrm{M}}$.*

(4) *$A \mathop{\smile\hspace{-0.9em}\mid}^{\text{þ}}_{C} B$ implies $B \mathop{\smile\hspace{-0.9em}\mid}^{\text{þ}}_{C} A$.*

(5) *$A \mathop{\smile\hspace{-0.9em}\mid}^{\text{þ}}_{C} B$ implies $B \mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{M}}_{C} A$.*

(6) *T admits a strict independence relation.*

Moreover, in a theory T satisfying these conditions, $\mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}$ is the coarsest strict independence relation.

In particular, T is rosy iff T^{eq} satisfies the equivalent conditions above.

Proof. First note that $\mathop{\smile\!\!\!\!\!\mid}^{\text{þ}} = \mathop{\smile\!\!\!\!\!\mid}^{\equiv\mathrm{M}*}$ by Proposition 2.36. Therefore we can apply Theorem 2.29: (1), (3) and (4) are equivalent, and they hold if and only if $\mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}$ is an independence relation. Moreover, they are equivalent to (2') $\mathop{\smile\!\!\!\!\!\mid}^{\equiv\mathrm{M}}$ satisfies the local character axiom, and to (5') $A \mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}_C B$ implies $B \mathop{\smile\!\!\!\!\!\mid}^{\equiv\mathrm{M}}_C A$.

The 'moreover' statement and the equivalence of (6) with the other conditions are by Theorem 1.30.

Finally, (1) $\implies$ (2) $\implies$ (2') and (4) $\implies$ (5) $\implies$ (5') since $A \mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}_C B \implies A \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{M}}_C B \implies A \mathop{\smile\!\!\!\!\!\mid}^{\equiv\mathrm{M}}_C B$ by Proposition 2.36 (3) and (4). ☐

Moreover, by Proposition 2.36 (2), if $\mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}$ is an independence relation then we also have an alternative characterisation of $\mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}$ by Theorem 2.27.

Corollary 2.38. *Every reduct of a rosy theory is rosy.*

Proof. Use condition (3) as in Corollary 2.33. ☐

Theorem 2.39. *The relation $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{M}}$ is a (strict) independence relation iff it is symmetric.*

Proof. If $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{M}}$ is symmetric, condition (5) of Theorem 2.37 is satisfied, and so $\mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}$ is an independence relation and therefore satisfies existence. Hence $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{M}}$ satisfies existence as well. Using symmetry and transitivity of $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{M}}$ it is easy to see that $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{M}}$ also satisfies extension, so $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{M}} = \mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}$. Conversely, if $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{M}}$ is an independence relation, then $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{M}}$ is symmetric by Theorem 1.14. ☐

The property of $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{M}}$ being symmetric is not stable under taking reducts:

Example 2.40. (Symmetry of $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{M}}$ is not preserved under reducts)
Let T_0 be the theory, in the signature of one unary function f, which states the following: there is at least one element; for every element b there are infinitely many elements a such that $f(a) = b$; and f has no periodic points. Note that T is complete.

Let T be the theory extending T_0, in the signature consisting of f and a binary relation E, stating that $\forall xy(Exy \leftrightarrow f(x) = y \vee f(y) = x)$. Then $A \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{M}}_C B \iff \operatorname{acl}(AC) \cap \operatorname{acl}(BC) = \operatorname{acl} C$, so $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{M}}$ is clearly symmetric. Yet the theory of an everywhere infinite forest from Example 1.31, for which $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{M}}$ is not symmetric, is a reduct of T.

Exercises

Exercise 2.41. (M-symmetry)
(i) Two algebraically closed sets A and B form a *modular pair* in the lattice of algebraically closed sets, written $\mathrm{M}(A,B)$, if the following rule holds: For any algebraically closed set $C \subseteq B$, $\mathrm{acl}(AC) \cap B = \mathrm{acl}(C(A \cap B))$. Show that $\mathrm{M}(A,B) \iff A \mathop{\smile\!\!\!\!\!\!\downarrow}^{\mathrm{M}}_{A\cap B} B$.
(ii) A lattice is called *M-symmetric* if $\mathrm{M}(A,B) \implies \mathrm{M}(B,A)$. Show that $\mathop{\smile\!\!\!\!\!\!\downarrow}^{\mathrm{M}}$ is symmetric iff the lattice of algebraically closed sets is M-symmetric.

Notes

The core of the results presented in this section is, of course, from Alf Onshuus. The entire development of the theory as presented here, however, is new. This is true, in particular, for the use of inconsistency pairs for thorn-forking, the definition of Ξ_{M}, and Theorem 2.39.

Exercise 2.41 is from [Sch96]. M-symmetric lattices are studied in a general context in [Ste99]. Note that a finite lattice is M-symmetric iff it is semimodular. Theorem 2.39 is a generalisation of [Sch96, Theorems 6.2.8 and 6.2.10], which state that $\mathop{\smile\!\!\!\!\!\!\downarrow}^{\mathrm{M}}$ is an independence relation if the lattice of algebraically closed subsets of the big model is M-symmetric and one of two additional conditions is satisfied. One of the additional conditions is simplicity of T. The other is strong atomicity of the lattice of algebraically closed sets (a condition that, in conjunction with M-symmetry, is roughly equivalent to the Steinitz exchange property for acl).

Example 2.40 is due to Wilfrid Hodges and first appeared in [EPP90]. It has become the standard example of a 1-based stable theory with a reduct that is not 1-based. It can also be found in [Pil96, Chapter 4, Example 6.1].

Chapter 3

Thorn-forking

This chapter covers some topics around the concepts of canonical bases in T^{eq}, both of types and of sequences of indiscernibles. It is *not* intended to be self-contained. It is based on Chapter 1 and a previous familiarity with simple theories, but it is completely independent of Chapter 2. The title is sort of justified by the fact that, as we will see, only thorn-forking independence can have canonical bases.

If I is a linearly ordered set, we occasionally write $\bar{a}_{\in I}$ for the I-sequence $(\bar{a}_i)_{i\in I}$. Sequences of indiscernibles can be indexed by an arbitrary *infinite* linearly ordered set I.

3.1 Canonical independence relations

Independence relations (on T^{eq}) that appear in the real world often have the following property:

Definition 3.1. *A relation* $\mathop{\smile\!\!\!\!\!\!|}$ *has the intersection property if it satisfies the following condition:*

(intersection)

Suppose $C_1 \subseteq B$ *and* $C_2 \subseteq B$ *are such that* $A \mathop{\smile\!\!\!\!\!\!|}_{C_1} B$ *and* $A \mathop{\smile\!\!\!\!\!\!|}_{C_2} B$*. Then* $A \mathop{\smile\!\!\!\!\!\!|}_{\operatorname{acl} C_1 \cap \operatorname{acl} C_2} B$*.*

An independence relation is canonical *if it is strict and has the intersection property.*

Note that if T is a simple theory with elimination of hyperimaginaries (such as a stable or supersimple theory), then $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ is a canonical independence relation for T^{eq}, because $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_{C} B$ holds for $C \subseteq B$ iff $\operatorname{cb}(\operatorname{stp}(\bar{a}/B)) \subseteq \operatorname{acl}^{\mathrm{eq}} C$. It is surprisingly easy to see that there can be at most one strict independence relation with this property:

Lemma 3.2. *Suppose $\mathop{\smile\!\!\!\!\!\!\!\perp}$ is a canonical independence relation. Then $\bar{a} \mathop{\smile\!\!\!\!\!\!\!\perp}_C B$ if and only if there is a sequence $(\bar{a}_i)_{i<\omega}$ of BC-indiscernibles that realise $\operatorname{tp}(\bar{a}/BC)$ and such that $\operatorname{acl}(C\bar{a}_{<k}) \cap \operatorname{acl}(C\bar{a}_{\geq k}) = \operatorname{acl} C$ for all $k < \omega$.*

Proof. First suppose that $\bar{a} \mathop{\smile\!\!\!\!\!\!\!\perp}_C B$. By Proposition 1.12 there is a $\mathop{\smile\!\!\!\!\!\!\!\perp}$-Morley sequence in $\operatorname{tp}(\bar{a}/BC)$ over C. By Proposition 1.9 and finite character, $\bar{a}_{<k} \mathop{\smile\!\!\!\!\!\!\!\perp}_C \bar{a}_{\geq k}$ for all $k < \omega$. Hence $C\bar{a}_{<k} \mathop{\smile\!\!\!\!\!\!\!\perp}_C C\bar{a}_{\geq k}$. By anti-reflexivity, $\operatorname{acl}(C\bar{a}_{<k}) \cap \operatorname{acl}(C\bar{a}_{\geq k}) = \operatorname{acl} C$. Note that for this direction it is only necessary that $\mathop{\smile\!\!\!\!\!\!\!\perp}$ is a strict independence relation.

Conversely, suppose there is a sequence $(\bar{a}_i)_{i<\omega}$ of BC-indiscernibles realising $\operatorname{tp}(\bar{a}/BC)$ and such that $\operatorname{acl}(C\bar{a}_{<k}) \cap \operatorname{acl}(C\bar{a}_{\geq k}) = \operatorname{acl} C$ for all $k < \omega$. Let κ be a regular cardinal number sufficiently big for the local character axiom. Define a totally ordered set $I = \kappa + \{*\} + \kappa'$, where κ' is a disjoint copy of κ having the opposite order, and $*$ is a new element greater than any element of κ and smaller than any element of κ'. Let $(\bar{a}_i)_{i\in I}$ be a BC-indiscernible extension of $(\bar{a}_i)_{i<\omega}$. Note that $\operatorname{acl}(C\bar{a}_{\in\kappa}) \cap \operatorname{acl}(C\bar{a}_{\in\kappa'}) = \operatorname{acl} C$.

By finite character and symmetry it is sufficient to prove that $\bar{a} \mathop{\smile\!\!\!\!\!\!\!\perp}_C \bar{b}$ for every finite tuple $\bar{b} \in B$. So let $\bar{b} \in B$ be a finite tuple.

By local character there is a subset $D \subseteq \bar{a}_{<\kappa}C$ such that $|D| < \kappa$ and $\bar{b} \mathop{\smile\!\!\!\!\!\!\!\perp}_D \bar{a}_{<\kappa}C$. By regularity of κ there is $\lambda < \kappa$ such that $D \subseteq \bar{a}_{<\lambda}C$. Therefore, by base monotonicity, $\bar{b} \mathop{\smile\!\!\!\!\!\!\!\perp}_{\bar{a}_{<\lambda}C} \bar{a}_{<\kappa}C$. Now using finite character it is easy to see that $\bar{b} \mathop{\smile\!\!\!\!\!\!\!\perp}_{\bar{a}_{<\lambda}C} \bar{a}_{\in I}C$. Hence, using base monotonicity again (and monotonicity), $\bar{b} \mathop{\smile\!\!\!\!\!\!\!\perp}_{\bar{a}_{\in\kappa}C} \bar{a}_*$.

Since the setup is symmetric with respect to reversing the order of I, $\bar{b} \mathop{\smile\!\!\!\!\!\!\!\perp}_{\bar{a}_{\in\kappa'}C} \bar{a}_*$ holds as well.

Applying the intersection property we get $\bar{b} \mathop{\smile\!\!\!\!\!\!\!\perp}_{\operatorname{acl} C} \bar{a}_*$. On the other hand $\bar{b} \mathop{\smile\!\!\!\!\!\!\!\perp}_C \operatorname{acl} C$ by extension. Applying symmetry to the last two statements, and then transitivity, we get $\bar{a}_* \mathop{\smile\!\!\!\!\!\!\!\perp}_C \bar{b}$, hence $\bar{a} \mathop{\smile\!\!\!\!\!\!\!\perp}_C \bar{b}$. $\square$

Theorem 3.3. *If $\mathop{\smile\!\!\!\!\!\!\!\perp}$ is a canonical independence relation, then $\mathop{\smile\!\!\!\!\!\!\!\perp} = \mathop{\smile\!\!\!\!\!\!\!\perp}^{\flat}$.*

Proof. Suppose $\mathop{\smile\!\!\!\!\!\!\!\perp}$ is a canonical independence relation. By Theorem 1.30 it follows that $\mathop{\smile\!\!\!\!\!\!\!\perp}^{\flat}$ is a (strict) independence relation, and it is the coarsest, so $\bar{a} \mathop{\smile\!\!\!\!\!\!\!\perp}_C B \implies \bar{a} \mathop{\smile\!\!\!\!\!\!\!\perp}^{\flat}_C B$. Therefore the only thing left to show is $\bar{a} \mathop{\smile\!\!\!\!\!\!\!\perp}^{\flat}_C B \implies \bar{a} \mathop{\smile\!\!\!\!\!\!\!\perp}_C B$. So suppose $\bar{a} \mathop{\smile\!\!\!\!\!\!\!\perp}^{\flat}_C B$. As in the first part of the proof of Lemma 3.2, there is a sequence $(\bar{a}_i)_{i<\omega}$ of BC-indiscernibles realising $\operatorname{tp}(\bar{a}/BC)$ and such that $\operatorname{acl}(C\bar{a}_{<k}) \cap \operatorname{acl}(C\bar{a}_{\geq k}) = \operatorname{acl} C$ for all $k < \omega$. Hence by Lemma 3.2, $\bar{a} \mathop{\smile\!\!\!\!\!\!\!\perp}_C B$. $\square$

There are rosy theories for which $\mathop{\smile\!\!\!\!\!\!\!\perp}^{\flat}$ is not canonical; cf. Section A.3 in the appendix.

Corollary 3.4. *Suppose T is simple and has elimination of hyperimaginaries (e. g., T is stable or supersimple). Then $\mathop{\smile\hspace{-0.9em}\vert}^{\mathrm{f}} = \mathop{\smile\hspace{-0.9em}\vert}^{\text{þ}}$, and this is the only strict independence relation for T^{eq}.*

Proof. If T is simple, $\mathop{\smile\hspace{-0.9em}\vert}^{\mathrm{f}}$ is a strict independence relation on T^{eq}, and types have canonical bases as hyperimaginary elements. If T also has elimination of hyperimaginaries, then types have canonical bases as sequences of imaginary elements. Therefore $\mathop{\smile\hspace{-0.9em}\vert}^{\mathrm{f}}$ satisfies the intersection property. Hence by the theorem, $\mathop{\smile\hspace{-0.9em}\vert}^{\mathrm{f}} = \mathop{\smile\hspace{-0.9em}\vert}^{\text{þ}}$. Since $\mathop{\smile\hspace{-0.9em}\vert}^{\mathrm{f}}$ is the finest strict independence relation for T^{eq} and $\mathop{\smile\hspace{-0.9em}\vert}^{\text{þ}}$ is the coarsest, $\mathop{\smile\hspace{-0.9em}\vert}^{\mathrm{f}} = \mathop{\smile\hspace{-0.9em}\vert}^{\text{þ}}$ is the only one. □

Exercises

Exercise 3.5. (independence in a reduct)

(i) Suppose T' is a reduct of T, $\mathop{\smile\hspace{-0.9em}\vert}$ is a strict independence relation for T, and $\mathop{\smile\hspace{-0.9em}\vert}'$ is a canonical independence relation for T'. Let A, B, C be subsets of the big model of T such that $C = \operatorname{acl} C$ and $A \mathop{\smile\hspace{-0.9em}\vert}_C B$ in T. Then $A \mathop{\smile\hspace{-0.9em}\vert}'_C B$.

(ii) Suppose T is simple and T' is a reduct of T that has elimination of hyperimaginaries. Let A, B, C be subsets of the big model of T such that $C = \operatorname{acl}^{\mathrm{eq}} C$ and $A \mathop{\smile\hspace{-0.9em}\vert}^{\mathrm{f}}_C B$ in T. Then $A \mathop{\smile\hspace{-0.9em}\vert}^{\mathrm{f}}_C B$ holds in T' as well.

Notes

Lemma 3.2 is from [Sch96, Theorem 1.7.3]. Theorem 3.3 is the result of combining [Sch96, Theorem 1.7.4] (which states that any canonical independence relation is the coarsest strict independence relation) with Theorem 1.30. The definition of the term ‘canonical independence relation’ in [Sch96] was via existence of weak canonical bases. Both definitions are in fact equivalent, as will be shown in Theorem 3.20.

$\mathop{\smile\hspace{-0.9em}\vert}^{\mathrm{f}} = \mathop{\smile\hspace{-0.9em}\vert}^{\text{þ}}$, the first statement of Corollary 3.4, is proved for stable theories in [Ons03a], and stated for simple theories with stable forking in [Ons03b]. The generalisation to simple theories with elimination of hyperimaginaries was first proved, in a different way, by Clifton Ealy.

3.2 Kernels of indiscernibles

Recall the convention that sequences of indiscernibles are infinite by definition.

Definition 3.6. *The* kernel *of a sequence of indiscernibles $(\bar{a}_i)_{i\in I}$ is the set*

$$\ker(\bar{a}_i)_{i\in I} = \{d \mid d \in \operatorname{dcl}(\bar{a}_{\in I}) \text{ and } (\bar{a}_i)_{i\in I} \text{ is indiscernible over } d\}.$$

The algebraic kernel *of a sequence of indiscernibles* $(\bar{a}_i)_{i\in I}$ *is the set*

$$\operatorname{aker}(\bar{a}_i)_{i\in I} = \bigl\{d \bigm| d \in \operatorname{acl}(\bar{a}_{\in I}) \text{ and } (\bar{a}_i)_{i\in I} \text{ is indiscernible over } d\bigr\}.$$

For the (algebraic) kernel computed in T^{eq} *we write* $\ker^{\mathrm{eq}}$ *and* $\operatorname{aker}^{\mathrm{eq}}$.

We will see below that $(\bar{a}_i)_{i\in I}$ is actually indiscernible over $\ker(\bar{a}_i)_{i\in I}$, so the kernel is the greatest set definable over the sequence over which it is indiscernible. Similarly, the algebraic kernel is the greatest set algebraic over the sequence over which it is indiscernible.

Definition 3.7. *Two sequences* $(\bar{a}_i)_{i\in I}$, $(\bar{a}_j)_{j\in J}$ *of indiscernibles are* cleanly collinear *if one of their concatenations* $(\bar{a}_i)_{i\in I+J}$ *and* $(\bar{a}_i)_{i\in J+I}$ *is indiscernible. ($I+J$ denotes the disjoint union ordered in such a way that $I<J$.) We write $\approx$ for the transitive closure of clean collinearity.*

Theorem 3.8. *Properties of the kernel:*

(1) *The kernel is a $\approx$-invariant:*
$(\bar{a}_i)_{i\in I} \approx (\bar{b}_j)_{j\in J} \implies \ker(\bar{a}_i)_{i\in I} = \ker(\bar{b}_j)_{j\in J}$.

(2) *If* $(\bar{a}_i)_{i\in I}$ *and* $(\bar{b}_j)_{j\in J}$ *are cleanly collinear, then* $\ker(\bar{a}_i)_{i\in I} = \operatorname{dcl}(\bar{a}_{\in I}) \cap \operatorname{dcl}(\bar{b}_{\in J})$.

(3) $\ker(\bar{a}_i)_{i\in I} = \operatorname{dcl}\ker(\bar{a}_i)_{i\in I}$.

(4) $(\bar{a}_i)_{i\in I}$ *is indiscernible over* $\ker(\bar{a}_i)_{i\in I}$.

Proof. (1) We may assume that $(\bar{a}_i)_{i\in I}$ and $(\bar{b}_j)_{j\in J}$ are cleanly collinear. Moreover, it suffices to prove $\ker(\bar{a}_i)_{i\in I} \subseteq \ker(\bar{b}_j)_{j\in J}$. So suppose $d \in \ker(\bar{a}_i)_{i\in I}$. Since $(\bar{b}_j)_{j\in J}$ is indiscernible over $\bar{a}_{\in I}$ and $d \in \operatorname{dcl}\bigl(\bar{a}_{\in I}\bigr)$, $(\bar{b}_j)_{j\in J}$ is indiscernible over d.

Since $d \in \operatorname{dcl}(\bar{a}_{\in I})$, there are $i_0 < \ldots < i_{k-1}$ in I and a formula $\varphi(\bar{x}_{<k}, \bar{y})$ such that $\varphi(\bar{a}_{i_0}, \ldots, \bar{a}_{i_{k-1}}, \bar{y})$ is a definition of d. Choose any $j_0 < \ldots < j_{k-1}$ in J. Then by indiscernibility of the concatenated sequence, $\varphi(\bar{b}_{j_0}, \ldots, \bar{b}_{j_{k-1}}, \bar{y})$ is also a definition of d. Therefore $d \in \operatorname{dcl}(\bar{b}_{\in J})$.

(2) $\ker(\bar{a}_i)_{i\in I} \subseteq \operatorname{dcl}(\bar{b}_{\in J})$ follows from (1) and the definition of the kernel. The opposite inclusion holds because $d \in \operatorname{dcl}(b_{\in J})$ implies that $(\bar{a}_i)_{i\in I}$ is indiscernible over d.

(3) It follows from (2) that the kernel can be written as the intersection of two dcl-closed sets, so it is itself closed under dcl.

(4) Let $(\bar{b}_j)_{j\in J}$ be cleanly collinear with $(\bar{a}_i)_{i\in I}$. Then $(\bar{a}_i)_{i\in I}$ is indiscernible over $(\bar{b}_j)_{j\in J}$, hence also over $\operatorname{dcl}\bigl((\bar{b}_j)_{j\in J}\bigr)$, hence also over $\ker(\bar{a}_i)_{i\in I} \subseteq \operatorname{dcl}\bigl((\bar{b}_j)_{j\in J}\bigr)$. $\square$

Remark 3.9. *If a sequence $(\bar{a}_i)_{i\in I}$ is indiscernible over a set B, then it is also indiscernible over* $\operatorname{acl} B$.

Proof. Suppose not. We may assume that I is such that we can extract a sequence from $(\bar{a}_i)_{i\in I}$ that is indiscernible over $\operatorname{acl} B$. Since the fact that the extracted sequence is indiscernible over $\operatorname{acl} B$ can be expressed by the type of the sequence over B, the original sequence must also be indiscernible over $\operatorname{acl} B$. □

Lemma 3.10. *In T^{eq}, the algebraic kernel is just the algebraic closure of the kernel:*

$$\operatorname{aker}^{\mathrm{eq}}(\bar{a}_i)_{i\in I} = \operatorname{acl}^{\mathrm{eq}} \ker^{\mathrm{eq}}(\bar{a}_i)_{i\in I}$$

Proof. Suppose $d \in \operatorname{acl}^{\mathrm{eq}} \ker^{\mathrm{eq}}(\bar{a}_i)_{i\in I}$. Let $\bar{d} \in \ker^{\mathrm{eq}}(\bar{a}_i)_{i\in I}$ be a finite tuple such that $d \in \operatorname{acl}^{\mathrm{eq}}(\bar{d})$. Then clearly $d \in \operatorname{acl}^{\mathrm{eq}}(\bar{d}) \subseteq \operatorname{acl}^{\mathrm{eq}}(\bar{a}_{\in I})$, and $(\bar{a}_i)_{i\in I}$ is indiscernible over $\bar{d}$ by Theorem 3.8 (4). Hence $(\bar{a}_i)_{i\in I}$ is indiscernible over $d \in \operatorname{acl}^{\mathrm{eq}}(\bar{d})$.

Conversely, suppose $d \in \operatorname{acl}^{\mathrm{eq}}(\bar{a}_{\in I})$ and $(\bar{a}_i)_{i\in I}$ is indiscernible over d. Let D be the set of conjugates of d over $\operatorname{acl}^{\mathrm{eq}}(\bar{a}_{\in I})$. Since D is finite, there is an element e coding it. (This is where we need T^{eq}.) Clearly $e \in \operatorname{dcl}^{\mathrm{eq}}(\bar{a}_{\in I})$. It is easy to see that $(\bar{a}_i)_{i\in I}$ is indiscernible over d' for every $d' \in D$. Hence $D \subseteq \ker^{\mathrm{eq}}\big((\bar{a}_i)_{i\in I}\big)$. By Theorem 3.8 (4), $(\bar{a}_i)_{i\in I}$ is indiscernible over D, hence also over e. Thus $e \in \ker^{\mathrm{eq}}(\bar{a}_i)_{i\in I})$ and $d \in \operatorname{acl}^{\mathrm{eq}}(e) \subseteq \operatorname{acl}^{\mathrm{eq}} \ker^{\mathrm{eq}}(\bar{a}_i)_{i\in I}$. □

Theorem 3.11. *Properties of the algebraic kernel:*

(1) *The algebraic kernel is a $\approx$-invariant:*
$(\bar{a}_i)_{i\in I} \approx (\bar{b}_j)_{j\in J} \implies \operatorname{aker}(\bar{a}_i)_{i\in I} = \operatorname{aker}(\bar{b}_j)_{j\in J}$.

(2) *If $(\bar{a}_i)_{i\in I}$ and $(\bar{b}_j)_{j\in J}$ are cleanly collinear,*
then $\operatorname{aker}(\bar{a}_i)_{i\in I} = \operatorname{acl}(\bar{a}_{\in I}) \cap \operatorname{acl}(\bar{b}_{\in J})$.

(3) $\operatorname{aker} \bar{a}_i)_{i\in I} = \operatorname{acl} \operatorname{aker}(\bar{a}_i)_{i\in I}$.

(4) $(\bar{a}_i)_{i\in I}$ *is indiscernible over* $\operatorname{aker}(\bar{a}_i)_{i\in I}$.

Proof. (1) It would be straightforward to prove this directly, in the same way as Theorem 3.11 (1). Instead we note the following equation:

$$\operatorname{aker}(\bar{a}_i)_{i\in I} = \mathcal{M} \cap \operatorname{aker}^{\mathrm{eq}}(\bar{a}_i)_{i\in I} = \mathcal{M} \cap \operatorname{acl}^{\mathrm{eq}} \ker^{\mathrm{eq}}(\bar{a}_i)_{i\in I}. \qquad (*)$$

By means this equation the statement is immediate from Theorem 3.8 (1).

(2) $\operatorname{aker}(\bar{a}_i)_{i\in I} \subseteq \operatorname{acl}(\bar{a}_{\in I})$ follows from the definition of the kernel. Analogously, $\operatorname{aker}(\bar{a}_i)_{i\in I} = \operatorname{aker}(\bar{b}_j)_{j\in J} \subseteq \operatorname{acl}(\bar{b}_{\in J})$. The opposite inclusion holds because $d \in \operatorname{acl}(b_{\in J})$ implies that $(\bar{a}_i)_{i\in I}$ is indiscernible over d.

(3) also follows immediately from $(*)$. (It is also straightforward to infer (3) from (2) as in Theorem 3.8 (3).)

(4) $(\bar{a}_i)_{i\in I}$ is indiscernible over $\ker^{\mathrm{eq}}(\bar{a}_i)_{i\in I}$ by Theorem 3.8. Hence $(\bar{a}_i)_{i\in I}$ is indiscernible over $\operatorname{aker}(\bar{a}_i)_{i\in I}$ by $(*)$. $\square$

Definition 3.12. *A set of indiscernibles $(\bar{a}_i)_{i\in I}$ is* based *on a set B if every automorphism σ of the big model that fixes B pointwise satisfies $\big(\sigma(\bar{a}_i)\big)_{i\in I} \approx (\bar{a}_i)_{i\in I}$. The set B is a* canonical base *for $(\bar{a}_i)_{i\in I}$ if the converse is also true.*

If a sequence of indiscernibles has a canonical base at all, then it is unique up to interdefinability. But the canonical base of a sequence of indiscernibles need not exist:

Example 3.13. (A sequence without a canonical base)
Let T be the theory of a dense linear order without endpoints. Consider the type $p\big((x_i)_{i\in\mathbb{Q}}\big) = \{x_i < x_j \mid i < j\}$. Now let $\big((a_i^n)_{i\in\mathbb{Q}}\big)_{n<\omega}$ be a sequence of sequences $(a_i^n)_{i\in\mathbb{Q}}$ realising this type, and more precisely $a_i^n < a_j^m$ iff either $i < j$, or $i = j$ and $n < m$. The sequence is indiscernible, and any two elements are equivalent under the type-definable equivalence relation $E\big((x_i)_{i\in\mathbb{Q}}, (y_i)_{i\in\mathbb{Q}}\big) = \{x_i < y_j \wedge y_i < x_j \mid i < j\}$. Therefore every automorphism of the big model that fixes the $\approx$-class of the sequence also fixes the E-class of the sequence. Yet $\ker\big((a_i^n)_{i\in\mathbb{Q}}\big)_{n<\omega} = \emptyset$. (Note that E also witnesses that T does not eliminate hyperimaginaries.)

Remark 3.14. *Let $(\bar{a}_i)_{i\in I}$ be an indiscernible sequence. Every automorphism σ of the big model that satisfies $\big(\sigma(\bar{a}_i)\big)_{i\in I} \approx (\bar{a}_i)_{i\in I}$ fixes* $\ker\big((\bar{a}_i)_{i\in I}\big)$ *pointwise.*

Proof. Suppose σ satisfies $\big(\sigma(\bar{a}_i)\big)_{i\in I} \approx (\bar{a}_i)_{i\in I}$. It is easy to see that there are sequences $(\bar{a}_i)_{i\in I} = (\bar{a}_i^{(0)})_{i\in I}, (\bar{a}_i^{(1)})_{i\in I}, \ldots, (\bar{a}_i^{(n)})_{i\in I} = \big(\sigma(\bar{a}_i)\big)_{i\in I}$ such that $(\bar{a}_i^{(j)})_{i\in I}$ and $(\bar{a}_i^{(j+1)})_{i\in I}$ are cleanly collinear. Let $\sigma^{(0)}$ be the identity automorphism of the big model, $\sigma^{(n)} = \sigma$ and $\sigma^{(j)}$ for $0 < j < n$ an arbitrary automorphism such that $\big(\sigma^{(j)}(\bar{a}_i)\big)_{i\in I} = (\bar{a}_i^{(j)})_{i\in I}$. It is clearly sufficient to prove the remark for the automorphisms $\sigma^{(j+1)} \circ (\sigma^{(j)})^{-1}$.

Therefore we may assume that $(\bar{a}_i)_{i\in I}$ and $\big(\sigma(\bar{a}_i)\big)_{i\in I}$ are cleanly collinear. Since the concatenated sequence is indiscernible over $\ker(\bar{a}_i)_{i\in I}$, there is an automorphism τ of the big model that fixes $\ker(\bar{a}_i)_{i\in I}$ pointwise and such that $\big(\tau(\bar{a}_i)\big)_{i\in I} = \big(\sigma(\bar{a}_i)\big)_{i\in I}$. Since σ and τ agree on $\bar{a}_{\in I}$, they also agree on $\ker(\bar{a}_i)_{i\in I} \subseteq \operatorname{dcl}(\bar{a}_{\in I})$. Thus σ also fixes $\ker(\bar{a}_i)_{i\in I}$ pointwise. $\square$

Corollary 3.15. *If an indiscernible sequence $(\bar{a}_i)_{i\in I}$ has a canonical base, then the canonical base is interdefinable with* $\ker(\bar{a}_i)_{i\in I}$.

Proof. Let $(\bar{a}_i)_{i\in I}$ be an indiscernible sequence that has a canonical base B. By Remark 3.14, every automorphism of the big model that fixes B pointwise also fixes $\ker(\bar{a}_i)_{i\in I}$ pointwise, so $\ker(\bar{a}_i)_{i\in I} \subseteq \operatorname{dcl} B$.

For the converse, first note that an automorphism fixing $\bar{a}_{\in I}$ also fixes $\bar{a}_{\in I}/\approx$, hence fixes B pointwise. Now consider a sequence $(\bar{b}_j)_{j\in J}$ that is cleanly collinear with $(\bar{a}_i)_{i\in I}$. Then $B \subseteq \operatorname{dcl}(\bar{a}_{\in I}) \cap \operatorname{dcl}(\bar{b}_{\in J})$, so $B \subseteq \ker(\bar{a}_i)_{i\in I}$. Therefore $\operatorname{dcl} B = \ker(\bar{a}_i)_{i\in I}$. □

Notes

The kernel was first defined by the author of [Sch96], who has changed his surname and his treatment of kernels since then. He now calls 'algebraic kernel' what he used to call 'kernel'. The canonical base of a sequence of indiscernibles was defined by Steven Buechler in [Bue97]. The relation of this notion to the kernel as stated in Corollary 3.15 is new.

Generalisation of this section (and, in fact, the entire chapter) to hyperimaginaries is essentially straightforward. A paper carrying this out is in preparation.

3.3 Weak canonical bases

In the following variant of Lemma 3.2 the hard direction is strengthened:

Lemma 3.16. *Suppose $\mathop{\smile\!\!\!\!\!\!\mid}$ is a canonical independence relation.*
Then $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_C B$ if and only if there is a sequence $(\bar{a}_i)_{i<\omega}$ of BC-indiscernibles that realise $\operatorname{tp}(\bar{a}/BC)$ and such that $\operatorname{acl}(\bar{a}_{<k}) \cap \operatorname{acl}(\bar{a}_{\geq k}) \subseteq \operatorname{acl} C$ for all $k < \omega$.

Proof. The 'only if' part is a weakening of the 'only if' part of Lemma 3.16.

Conversely, suppose there is a sequence $(\bar{a}_i)_{i<\omega}$ of BC-indiscernibles realising $\operatorname{tp}(\bar{a}/BC)$ and such that $\operatorname{acl}(\bar{a}_{<k}) \cap \operatorname{acl}(\bar{a}_{\geq k}) \subseteq \operatorname{acl} C$ for all $k < \omega$. Let $(\kappa = |T| + |BC|)^+$. Define a totally ordered set $I = \kappa + \{*\} + \kappa'$, where κ' is a disjoint copy of κ having the opposite order, and $*$ is a new element greater than any element of κ and smaller than any element of κ'. Let $(\bar{a}_i)_{i\in I}$ be a BC-indiscernible extension of $(\bar{a}_i)_{i<\omega}$. Note that $\operatorname{acl}(\bar{a}_{\in\kappa}) \cap \operatorname{acl}(\bar{a}_{\in\kappa'}) \subseteq \operatorname{acl} C$.

By symmetry it is sufficient to prove that $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_C B$.

By extended local character (see Exercise 1.6 (ii)) there is a subset $D \subseteq \bar{a}_{<\kappa}$ such that $|D| < \kappa$ and $BC \mathop{\smile\!\!\!\!\!\!\mid}_D \bar{a}_{<\kappa}$. By regularity of κ there is $\lambda < \kappa$ such that $D \subseteq \bar{a}_{<\lambda}$. Therefore, by base monotonicity, $BC \mathop{\smile\!\!\!\!\!\!\mid}_{\bar{a}_{<\lambda}} \bar{a}_{<\kappa}$. Now using finite character it is easy to see that $BC \mathop{\smile\!\!\!\!\!\!\mid}_{\bar{a}_{<\lambda}} \bar{a}_{\in I}$. Hence, using base monotonicity again (and monotonicity), $BC \mathop{\smile\!\!\!\!\!\!\mid}_{\bar{a}_{\in\kappa}} \bar{a}_*$.

Since the setup is symmetric with respect to reversing the order of I, $BC \mathop{\smile\!\!\!\!\!\!\mid}_{\bar{a}_{\in\kappa'}} \bar{a}_*$ holds as well.

Applying the intersection property we get $BC \mathop{\smile\!\!\!\!\!\!\mid}_D \bar{a}_*$, where D is defined as $D = \operatorname{acl}(\bar{a}_{\in\kappa}) \cap \operatorname{acl}(\bar{a}_{\in\kappa'}) \subseteq \operatorname{acl} C$. By symmetry, $\bar{a}_* \mathop{\smile\!\!\!\!\!\!\mid}_D BC$, and by extension, $\bar{a}_* \mathop{\smile\!\!\!\!\!\!\mid}_D B \operatorname{acl} C$. Hence $\bar{a}_* \mathop{\smile\!\!\!\!\!\!\mid}_{\operatorname{acl} C} B$ by base monotonicity and monotonicity.

On the other hand $\bar{a}_* \mathop{\smile\!\!\!\!\!\!\mid}_C \operatorname{acl} C$ by extension. Applying symmetry to the last two statements, and then transitivity, we get $B \mathop{\smile\!\!\!\!\!\!\mid}_C \bar{a}_*$, hence $B \mathop{\smile\!\!\!\!\!\!\mid}_C \bar{a}$, hence $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_C B$. □

The following theorem is essentially just a more conceptual re-formulation of Lemma 3.16:

Theorem 3.17. *Let* $\mathop{\smile\!\!\!\!\!\!\mid}$ *be a canonical independence relation.*
Then $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_C B$ *holds if and only if there is a sequence* $(\bar{a}_i)_{i<\omega}$ *of indiscernibles realising* $\operatorname{tp}(\bar{a}/BC)$ *such that* $\operatorname{aker}(\bar{a}_i)_{i<\omega} \subseteq \operatorname{acl} C$.

Proof. To reduce the theorem to Lemma 3.16, it suffices to prove that for a BC-indiscernible sequence $(\bar{a}_i)_{i<\omega}$, $\operatorname{aker}(\bar{a}_i)_{i<\omega} \subseteq \operatorname{acl} C$ holds if and only if $\operatorname{acl}(\bar{a}_{<k}) \cap \operatorname{acl}(\bar{a}_{\geq k}) \subseteq \operatorname{acl} C$ holds for all $k < \omega$.

For the 'if' part of the claim, suppose $\operatorname{aker}\big((\bar{a}_i)_{i<\omega}\big) \subseteq \operatorname{acl} C$ holds. Let I be a totally ordered set and $J = I + \omega$. Let $(\bar{a}_j)_{j\in J}$ be a BC-indiscernible extension of $(\bar{a}_i)_{i<\omega}$. Then for every $k < \omega$, $(\bar{a}_i)_{i\in J, i<k}$ and $(\bar{a}_i)_{i\in J, i\geq k}$ are cleanly collinear sequences of BC-indiscernibles. Hence $\operatorname{acl}(\bar{a}_{<k}) \cap \operatorname{acl}(\bar{a}_{\geq k}) \subseteq \operatorname{aker}(\bar{a}_i)_{i\in J} \subseteq \operatorname{acl} C$.

For the 'only if' part of the claim, suppose $\operatorname{acl}(\bar{a}_{<k}) \cap \operatorname{acl}(\bar{a}_{\geq k}) \subseteq \operatorname{acl} C$ holds for all $k < \omega$ and consider an arbitrary element $d \in \operatorname{aker} \bar{a}_i)_{i<\omega}$ of the algebraic kernel. Let $\varphi(\bar{x}_{<k}, y)$ be such that $\varphi(\bar{a}_{<k}, y)$ is an algebraic formula realised by d. Since $(\bar{a}_i)_{i<\omega}$ is indiscernible over d, $\varphi(\bar{a}_{k+1}, \ldots, \bar{a}_{2k}, y)$ is also an algebraic formula realised by d. Hence $d \in \operatorname{acl}(\bar{a}_{<k}) \cap \operatorname{acl}(\bar{a}_{\geq k}) \subseteq \operatorname{acl} C$. □

Corollary 3.18. *Let* $\mathop{\smile\!\!\!\!\!\!\mid}$ *be a canonical independence relation.*
A sequence $(\bar{a}_i)_{i<\omega}$ *of* C*-indiscernibles is a* $\mathop{\smile\!\!\!\!\!\!\mid}$*-Morley sequence for* $\operatorname{tp}(\bar{a}_0/C)$ *if and only if* $\operatorname{aker}(\bar{a}_i)_{i<\omega} \subseteq \operatorname{acl} C$.

Proof. First suppose $\operatorname{aker}(\bar{a}_i)_{i<\omega} \subseteq \operatorname{acl} C$. Then $\bar{a}_0 \mathop{\smile\!\!\!\!\!\!\mid}_C \{\bar{a}_i \mid 0 < i < \omega\}$ by Theorem 3.17, so $(\bar{a}_i)_{i<\omega}$ is $\mathop{\smile\!\!\!\!\!\!\mid}$-independent over C, hence a $\mathop{\smile\!\!\!\!\!\!\mid}$-Morley sequence over C. Conversely, suppose $(\bar{a}_i)_{i<\omega}$ is a $\mathop{\smile\!\!\!\!\!\!\mid}$-Morley sequence over C. Consider a C-indiscernible extension $(\bar{a}_i)_{i<\omega+\omega}$ of $(\bar{a}_i)_{i<\omega}$. Then $\bar{a}_{<\omega} \mathop{\smile\!\!\!\!\!\!\mid}_C \{\bar{a}_i \mid \omega \leq i < \omega + \omega\}$, so $\operatorname{aker}(\bar{a}_i)_{i<\omega} \subseteq \operatorname{acl} C$ by anti-reflexivity of $\mathop{\smile\!\!\!\!\!\!\mid}$. □

Definition 3.19. *Let* $\mathop{\smile\!\!\!\!\!\!\mid}$ *be an independence relation,* $\bar{a}$ *a tuple and* $B = \operatorname{acl} B$ *an algebraically closed set. An algebraically closed set* $C \subseteq B$ *is called the* weak canonical base *(with respect to* $\mathop{\smile\!\!\!\!\!\!\mid}$*) of* $\operatorname{tp}(\bar{a}/B)$ *if* C *is the smallest algebraically closed subset of* B *such that* $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_C B$.

We say that $\mathop{\smile\!\!\!\!\!\!|}$ has weak canonical bases *if* $\operatorname{tp}(\bar{a}/B)$ *has a weak canonical base for every tuple* $\bar{a}$ *and every algebraically closed set* B.

Thus $C = \operatorname{acl} C \subseteq B = \operatorname{acl} B$ is the weak canonical base of $\operatorname{tp}(\bar{a}/B)$ iff $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_C B$ holds, and for every $D = \operatorname{acl} D \subseteq B$ such that $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_D B$ we have $C \subseteq D$. If a type has a weak canonical base, then it is of course unique.

Theorem 3.20. *A strict independence relation has weak canonical bases if and only if it is canonical.*

Moreover, if $\mathop{\smile\!\!\!\!\!\!|}$ *is a canonical independence relation and* B *is an algebraically closed set, then the weak canonical base of a type* $\operatorname{tp}(\bar{a}/B)$ *is* $\operatorname{aker}(\bar{a}_i)_{i<\omega}$*, where* $(\bar{a}_i)_{i<\omega}$ *is an arbitrary* $\mathop{\smile\!\!\!\!\!\!|}$*-Morley sequence for* $\operatorname{tp}(\bar{a}/B)$.

Proof. First suppose $\mathop{\smile\!\!\!\!\!\!|}$ has weak canonical bases and $C_1 \subseteq B$ and $C_2 \subseteq B$ are such that $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_{C_1} B$ and $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_{C_1} B$. It easily follows that $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_{\operatorname{acl} C_1} \operatorname{acl} B$ and $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_{\operatorname{acl} C_1} \operatorname{acl} B$ also hold. Thus if C is the weak canonical base of $\operatorname{tp}(\bar{a}/\operatorname{acl} B)$, then $C \subseteq \operatorname{acl} C_1$ and $C \subseteq \operatorname{acl} C_2$. Hence $C \subseteq \operatorname{acl} C_1 \cap \operatorname{acl} C_2$, and so $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_{\operatorname{acl} C_1 \cap \operatorname{acl} C_2} \operatorname{acl} B$ by base monotonicity.

Conversely, suppose $\mathop{\smile\!\!\!\!\!\!|}$ is a canonical independence relation. Let B be an algebraically closed set and $(\bar{a}_i)_{i<\omega}$ a $\mathop{\smile\!\!\!\!\!\!|}$-Morley sequence for $\operatorname{tp}(\bar{a}/B)$. Then $C = \operatorname{aker}(\bar{a}_i)_{i<\omega}$ is a weak canonical base for $\operatorname{tp}(\bar{a}/B)$ (note that this also proves the 'moreover' part):

First we observe that in fact $C \subseteq B$ by Corollary 3.18. It follows from Theorem 3.17 that $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_C B$. Finally we need to prove minimality of C. So suppose $C' \subseteq B$ and $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_{C'} B$. Then $(\bar{a}_i)_{i<\omega}$ is a $\mathop{\smile\!\!\!\!\!\!|}$-Morley sequence over C'. Hence by Corollary 3.18, $C = \operatorname{aker}(\bar{a}_i)_{i<\omega} \subseteq \operatorname{acl} C'$ holds. □

Exercises

Exercise 3.21. (weak canonical bases)

Suppose $\mathop{\smile\!\!\!\!\!\!|}$ is a canonical independence relation. Given a tuple $\bar{a}$ and an algebraically closed set B, let $\operatorname{wcb}(\bar{a}/B)$ denote the weak canonical base of $\operatorname{tp}(\bar{a}/B)$.

(i) $\bar{a} \equiv_B \bar{a}'$ implies $\operatorname{wcb}(\bar{a}/\operatorname{acl} B) = \operatorname{wcb}(\bar{a}'/\operatorname{acl} B)$. Hence it is in no way misleading to define $\operatorname{wcb}(\bar{a}/B) = \operatorname{wcb}(\bar{a}/\operatorname{acl} B)$ if B is not algebraically closed.

(ii) If $\bar{a}'$ is a subtuple of $\bar{a}$, then $\operatorname{wcb}(\bar{a}'/B) \subseteq \operatorname{wcb}(\bar{a}/B)$.

(iii) The following conditions are equivalent for a tuple $\bar{a}$ and sets B, C:
(1) $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_C B$, (2) $\operatorname{wcb}(\bar{a}/BC) \subseteq \operatorname{acl} C$, (3) $\operatorname{wcb}(\bar{a}/BC) = \operatorname{wcb}(\bar{a}/C)$.

Notes

Theorem 3.20 is new. Weak canonical bases in the sense of this section were defined in [Sch96]. They first appeared, with a different definition, in [EPP90].

The definition in [EPP90] implies stability, and both definitions are equivalent for stable theories.

3.4 Canonical bases in simple theories

If T is a stable theory or, more generally, a simple theory with elimination of hyperimaginaries, then forking independence is a canonical independence relation for T^{eq} because an amalgamation base $\mathrm{tp}(\bar{a}/B)$ does not fork over a subset $C \subseteq B$ if and only if $\mathrm{cb}(\bar{a}/B) \subseteq \mathrm{acl}^{\mathrm{eq}} C$, and so $\mathrm{acl}^{\mathrm{eq}} \mathrm{cb}(\bar{a}/B)$ is the weak canonical base of $\mathrm{tp}(\bar{a}/B)$. Hence $\mathrm{cb}(\bar{a}/B)$ is characterised by Theorem 3.20 up to interalgebraicity. (Note that $\mathrm{cb}(\bar{a}/B)$ is only defined up to interdefinability. So it makes sense to regard $\mathrm{cb}(\bar{a}/B)$ as a definably closed set.) In this section we will try to improve this result. We will also see that the phenomenon exhibited by Example 3.13 cannot occur in this context.

We will need the well-known fact that the type of an indiscernible sequence over a cleanly collinear sequence is an amalgamation base:

Remark 3.22. *Suppose $\bar{a}_{\in I}$ and $\bar{b}_{\in J}$ are cleanly collinear sequences of indiscernibles. Then $\mathrm{tp}(\bar{a}_{\in I}/\bar{b}_{\in J})$ is a strong type.*

Proof. We may assume that the concatenation $\bar{a}_{\in I}{}^\frown\bar{b}_{\in J}$ is indiscernible (otherwise exchange $\bar{a}_{\in I}$ and $\bar{b}_{\in J}$). Suppose $\bar{a}'_{\in I}$ realises $\mathrm{tp}(\bar{a}_{\in I}/\bar{b}_{\in J})$. Consider a finite equivalence relation $\varphi(\bar{x}_0 \ldots \bar{x}_{n-1}, \bar{x}'_0 \ldots \bar{x}'_{n-1})$ definable over $\bar{b}_{\in J}$. We need to show that $\models \varphi(\bar{a}_{i_0} \ldots \bar{a}_{i_{n-1}}, \bar{a}'_{i_0} \ldots \bar{a}'_{i_{n-1}})$ for any $i_0 < \cdots < i_{n-1}$ in I.

Note that by indiscernibility all tuples $\bar{a}_{i_0} \ldots \bar{a}_{n-1}$ ($i_0 < \cdots < i_{n-1}$ in I) must be equivalent, since otherwise all would have to be inequivalent, which is impossible because φ has only finitely many classes. By compactness there is a sequence $\bar{c}_{<\omega}$ such that $\bar{a}_{\in I}{}^\frown\bar{c}_{<\omega}{}^\frown\bar{b}_{\in J}$ and $\bar{a}_{\in I}{}^\frown\bar{c}_{<\omega}{}^\frown\bar{b}_{\in J}$ are both indiscernible. Now clearly $\bar{a}_{i_0} \ldots \bar{a}_{i_{n-1}}$, $\bar{c}_0 \ldots \bar{c}_{n-1}$ and $\bar{a}'_{i_0} \ldots \bar{a}'_{i_{n-1}}$ are φ-equivalent. $\square$

For the rest of this section we work in T^{eq} for a fixed simple theory T with elimination of hyperimaginaries (EHI), and we freely use well-known facts about simple theories (cf. [Kim96], [KP97] and [HKP00]).

Lemma 3.23. *(T simple with EHI)*
Let $\bar{a}_{\in I}$ and $\bar{b}_{\in J}$ be cleanly collinear sequences of indiscernibles and $C = \mathrm{cb}(\bar{a}_{\in I}/\bar{b}_{\in J})$. If $\bar{a}'_{\in I} \equiv_C \bar{a}_{\in I}$, then $\bar{a}'_{\in I} \approx \bar{a}_{\in I}$.

Proof. Let $\bar{b}'_{\in J}$ be such that $\bar{a}_{\in I}\bar{b}_{\in J} \equiv_C \bar{a}'_{\in I}\bar{b}'_{\in J}$. Without loss of generality $\bar{b}_{\in J} \mathop{\smile\!\!\!\!\!\!\mid}_C \bar{b}'_{\in J}$ (otherwise we can find $\bar{a}''_{\in I}\bar{b}''_{\in J} \equiv_C \bar{a}_{\in I}\bar{b}_{\in J}$ independent from

$\bar{b}_{\in J}\bar{b}'_{\in J}$ over C). Note that $\mathrm{tp}(\bar{a}_{\in I}/\bar{b}_{\in J})$ and $\mathrm{tp}(\bar{a}'_{\in I}/\bar{b}'_{\in J})$ are non-forking extensions of the same amalgamation base $\mathrm{tp}(\bar{a}_{\in I}/\bar{b}_{\in J})$. By the amalgamation theorem for amalgamation bases there is a type $\mathrm{tp}(\bar{c}_{\in I}/\bar{b}_{\in J}\bar{b}'_{\in J})$ which is a common non-forking extension of both. Since $\bar{c}_{\in I}$ is cleanly collinear with both $\bar{b}_{\in J}$ and $\bar{b}'_{\in J}$, we have $\bar{a}_{\in I} \approx \bar{b}_{\in J} \approx \bar{c}_{\in I} \approx \bar{b}'_{\in J} \approx \bar{a}'_{\in I}$. □

Remark 3.24. *(T simple with EHI)*
If $\bar{b}_{\in J}$ is indiscernible over $\bar{a}$, then the relation $\mathrm{cb}(\bar{a}/\bar{b}_{\in J}) \subseteq \ker \bar{b}_{\in J}$ holds.

Proof. Let $\bar{c}_{<\omega}$ be cleanly collinear with $\bar{b}_{\in J}$ over $\bar{a}$. Then $\mathrm{cb}(\bar{a}/\bar{b}_{\in J}) = \mathrm{cb}(\bar{a}/\bar{c}_{<\omega})$ is definable over $\bar{b}_{\in J}$ and over $\bar{c}_{\in K}$, hence $\mathrm{cb}(\bar{a}/\bar{b}_{\in J}) \subseteq \mathrm{dcl}\, \bar{b}_{\in J} \cap \mathrm{dcl}\, \bar{c}_{<\omega} = \ker \bar{b}_{\in J}$. □

Putting together the last two results and Remark 3.14, we get something quite close to the desired refinement of Theorem 3.20:

Theorem 3.25. *(T simple with EHI)*
If $\bar{a}_{\in I}$ and $\bar{b}_{\in J}$ are cleanly collinear sequences of indiscernibles, then

$$\mathrm{cb}(\bar{a}_{\in I}/\bar{b}_{\in J}) = \ker \bar{a}_{\in I},$$

and this is a canonical base for $\bar{a}_{\in I}$ in the sense of Definition 3.12.

Proof. An automorphism σ of the big model that fixes pointwise $\ker a_{\in I}$ fixes $\mathrm{cb}(\bar{a}_{\in I}/\bar{b}_{\in J})$ pointwise by Remark 3.24. An automorphism σ that fixes pointwise $\mathrm{cb}(\bar{a}_{\in I}/\bar{b}_{\in J})$ satisfies $\sigma(\bar{a}_{\in I}) \approx \bar{a}_{\in I}$ by Lemma 3.23. An automorphism σ that satisfies $\sigma(\bar{a}_{\in I}) \approx \bar{a}_{\in I}$ fixes $\ker a_{\in I}$ pointwise by Remark 3.14. □

Corollary 3.26. *(T simple with EHI)*
Let $p(\bar{x})$ be an amalgamation base. If $\bar{a}_{<\omega}$ is a Morley sequence for $p(\bar{x})$, then

$$\mathrm{cb}(p) \subseteq \ker(\bar{a}_{<\omega}) \subseteq \mathrm{acl}\, \mathrm{cb}(p).$$

Proof. By compactness there is a sequence $\bar{b}_{<\omega}$ such that the concatenation $\bar{b}_{<\omega}{}^\frown\bar{a}_{<\omega}$ is still a Morley sequence for p. Note that $\mathrm{tp}(\bar{b}_{<\omega}/\bar{a}_{<\omega})$ is an amalgamation base by Remark 3.22. Hence $\mathrm{cb}(p) = \mathrm{cb}(\bar{b}_0/\bar{a}_{<\omega}) \subseteq \mathrm{cb}(\bar{b}_{<\omega}/\bar{a}_{<\omega}) = \ker(\bar{a}_{\in I})$.

For the inclusion on the right-hand side just observe that $\ker(\bar{a}_{<\omega}) \subseteq \mathrm{aker}(\bar{a}_{<\omega}) = \mathrm{acl}\, \mathrm{cb}(p)$ because $\mathrm{aker}(\bar{a}_{<\omega})$ is the weak canonical base of p by Theorem 3.20. □

I would have liked to show that $\mathrm{cb}\big(\mathrm{stp}(\bar{a}/B)\big) = \ker \bar{a}_{<\omega}$, but here is a (perfectly trivial 1-based supersimple) counter-example:

Example 3.27. (Alzheimer's random graph)
Consider the following theory T: There are two sorts N and C and a partial function $f : N \times N \to C$. C has precisely 2 elements ('edge' and 'no edge'). $f(a, b)$ is defined iff $a \neq b$. If we fix an element $e \in C$, the relation $f(x, y) = e$ defines a random graph on N. Note that $\operatorname{dcl}\emptyset = \emptyset$, while $\operatorname{acl}\emptyset = C$. Like the theory of the random graph T is supersimple.

We have $\operatorname{tp}(a/\emptyset) \vdash \operatorname{stp}(a/\emptyset)$ for every single element $a \in N$. For distinct $a, b \in N$, however, we have $\operatorname{tp}(ab/\emptyset) \nvdash \operatorname{stp}(ab/\emptyset)$ because $\operatorname{stp}(ab/\emptyset)$ fixes $f(a, b)$ while $\operatorname{tp}(ab/\emptyset)$ does not.

Hence for any Morley sequence $(a_i)_{i<\omega+2}$ in $\operatorname{tp}(a/\emptyset)$ we have $\operatorname{cb}(a_\omega/a_{<\omega}) = \emptyset$ while $\operatorname{cb}(a_\omega a_{\omega+1}/a_{<\omega}) = C = \ker(a_{<\omega})$.

Yet it turns out that the stronger statement does hold for stable theories:

Corollary 3.28. *(T stable)*
Let $p(\bar{x})$ be a stationary type. If $\bar{a}_{<\omega}$ is a Morley sequence for $p(\bar{x})$, then

$$\operatorname{cb}(p) = \ker \bar{a}_{<\omega}.$$

Proof. It is sufficient to show that $\operatorname{cb}(\bar{b}_0/\bar{a}_{<\omega}) = \operatorname{cb}(\bar{b}_{<\omega}/\bar{a}_{<\omega})$ holds in the proof of Corollary 3.26. Let $C = \operatorname{cb}(\bar{b}_0/\bar{a}_{<\omega})$. Since clearly $C \subseteq \operatorname{cb}(\bar{b}_{<\omega}/\bar{a}_{<\omega})$ we need only prove $\operatorname{cb}(\bar{b}_{<\omega}/\bar{a}_{<\omega}) \subseteq C$. $\bar{b}_{<\omega} \mathop{\smile\!\!\!\!\!\!|}_C \bar{a}_{<\omega}$ holds by a standard forking calculation, so the only thing left to show is that $\operatorname{tp}(\bar{b}_{<\omega}/C)$ is stationary. But this is true because $\operatorname{tp}(\bar{a}_i/C)$ is stationary for every $i < \omega$ and $\bar{a}_{<\omega}$ is independent over C. $\square$

Exercises

Exercise 3.29. (1-based theories, cf. Exercise 1.7)
Let T be a simple theory with elimination of hyperimaginaries.
(i) T is called 1-based if $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_{\operatorname{acl}^{\mathrm{eq}} A \cap \operatorname{acl}^{\mathrm{eq}} B} B$ holds for all A, B. Show that T is 1-based iff the lattice of algebraically closed sets in the big model of T^{eq} is modular.
(ii) Show that T is 1-based and perfectly trivial iff the lattice of algebraically closed sets in the big model of T^{eq} is distributive.

Notes

Exercise 3.29 is from [Sch96], the rest is new. The entire chapter can be generalised to hyperimaginaries—a paper containing the details is in preparation.

Appendix A

Appendix

A.1 Thorn-forking—the official definition

Thorn-forking (or þ-forking) is a notion of independence first defined by Alf Onshuus as the notion of independence corresponding to certain ranks (the thorn-ranks or þ-ranks) suggested by Thomas Scanlon. In this section I present the original definition and prove that it is equivalent to the definition in Section 1.5 if it is read in T^{eq} (as is the original definition).

Therefore we work in T^{eq} throughout. Here is the definition from [Ons03a, Definition 2.1]:

Definition A.1. *Let $\varphi(x,y)$ be an $\mathcal{L}$-formula without parameters, let b be an element, and let C be a set.*

- *The formula $\varphi(x,b)$ is said to* strongly divide *over C if $\mathrm{tp}(b/C)$ is not algebraic and $\{\varphi(x,b') \mid b' \models \mathrm{tp}(b/C)\}$ is k-inconsistent for some natural number $k < \omega$.*
- *The formula $\varphi(x,b)$ is said to* þ-divide *over C if there is a tuple c such that $\varphi(x,b)$ strongly divides over Cc.*
- *The formula $\varphi(x,b)$ is said to* þ-fork *over C if it implies a (finite) disjunction of formulas (with arbitrary parameters), each of which þ-divides over C.*

Let us begin with some easy observations about these definitions:

Following Onshuus, we never mentioned identifying logically equivalent formulas (so we will not do it) or required that b be a canonical parameter of $\varphi(x,b)$. Thus, if $\varphi(x,b)$ strongly divides over C, z is a variable in a non-algebraic sort, $\psi(x,y,z)$ is the formula $\varphi(x,y) \wedge z = z$, and c is an element

of the same sort as z, then even though $\varphi(x,b)$ and $\psi(x,bc)$ are equivalent, $\psi(x,bc)$ does not strongly divide over C. More generally, a formula containing an unused parameter of a non-algebraic sort can never þ-divide over any set.

Using compactness, one can prove that a formula $\varphi(x,b)$ strongly divides over C if and only if the set $\{\varphi(x,b') \mid b' \models \operatorname{tp}(b/C)\}$ of its C-conjugates is infinite and has no infinite consistent subset. (This is [Ons03a, Remark 2.1.1].)

Again using compactness, it is easy to see that in the definition of þ-dividing it does not matter whether we demand that c be a *finite* tuple. In fact, a formula $\varphi(x,b)$ þ-divides over a set C if and only if it strongly divides over a superset $C' \supseteq C$ of C.

Proposition A.2. *Our new definition of þ-forking ($\mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}$ as defined in Definition 1.26) agrees with the original definition by Alf Onshuus:*

$A \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_C B$ if and only if for every tuple $a \in A$ and every tuple $b \in BC$ and $\mathcal{L}$-formula $\varphi(x,y)$ such that $\models \varphi(a,b)$, the formula $\varphi(x,b)$ does not þ-fork over C.

Proof. We first prove the implication from left to right, assuming that for some $a \in A$ a formula in the type $\operatorname{tp}(a/BC)$ þ-forks over C. So we have $\operatorname{tp}(a/BC) \models \varphi_1(x,b_1) \vee \ldots \vee \varphi_n(x,b_n)$, where each $\varphi_i(x,b_i)$ strongly divides over some superset $C_i \supseteq C$. Now choosing $\hat{B} = BC_1 \ldots C_n b_1 \ldots b_n$ we can demonstrate that $A' \not\mathop{\smile\!\!\!\!\!\!|}^{\text{M}}_C \hat{B}$ for all $A' \equiv_{BC} A$:

For any $A' \equiv_{BC} A$ there is $a' \equiv_{BC} a$, $a' \in A'$, and i such that $\models \varphi_i(a',b_i)$ holds. Since $\varphi_i(x,b_i)$ strongly divides over C_i, we have the following: $b_i \notin \operatorname{acl} C_i$, but only finitely many realisations $b_i' \models \operatorname{tp}(b_i/C_i)$ can simultaneously satisfy $\varphi_i(a',b_i')$. Thus $b_i \in (\operatorname{acl}(C_i A') \cap \operatorname{acl} \hat{B}) \setminus \operatorname{acl} C_i$, so $A' \not\mathop{\smile\!\!\!\!\!\!|}^{\text{M}}_C \hat{B}$.

We will now prove the implication from right to left in two steps.

First we claim the following: Suppose $B = \operatorname{acl}(BC)$. If there is no $a \in A$ s.t. a formula in $\operatorname{tp}(a/BC)$ þ-forks over C, then $A \mathop{\smile\!\!\!\!\!\!|}^{\text{M}}_C B$.

Suppose $A \not\mathop{\smile\!\!\!\!\!\!|}^{\text{M}}_C B$. Then there is a set C' satisfying $C \subseteq C' \subseteq B$ such that $\operatorname{acl}(AC') \cap B \supsetneq \operatorname{acl} C'$. Let $b \in \operatorname{acl}(AC') \cap B \setminus \operatorname{acl} C'$ witness this. Let $a \in A$, $c \in C'$ and $\varphi(x,y,z)$ be such that the formula $\varphi(a,y,c)$ is algebraic (has at most k realisations, say) and realised by b. Define $\varphi'(x,y,z)$ as follows: $\varphi(x,y,z) \wedge \exists_{\leq k} y \varphi(x,y,z)$. Then $\models \varphi'(a,b,c)$ also holds. The formula $\varphi'(x,b,c)$ strongly divides over C' since $bc \notin \operatorname{acl} C'$ (because $b \notin \operatorname{acl} C'$) and every consistent subset of $\left\{\varphi'(x,b',c') \;\middle|\; b'c' \models \operatorname{tp}(bc/C')\right\}$ has at most k elements. Since $\varphi'(x,b,c) \in \operatorname{tp}(a/B)$ it follows that $A \not\mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_C B$.

Now we can finish the proof. Suppose there is no $a \in A$ such that a formula in $\operatorname{tp}(a/BC)$ þ-forks over C, and consider any $\hat{B} \supseteq B$. By [Ons03a, Lemma 2.1.2(1)] (the proof of which works for infinite sets as well as for finite tuples) there is $A' \equiv_{BC} A$ such that when $\bar{a}'$ enumerates A', $\operatorname{tp}(\bar{a}'/\operatorname{acl}(\hat{B}C))$

does not þ-fork over C. By the claim this implies that $A' \mathop{\smile\!\!\!\!\!\!\;\mid}^{\mathrm{M}}_C \operatorname{acl}(\hat{B}C)$, so we get $A' \mathop{\smile\!\!\!\!\!\!\;\mid}^{\mathrm{M}}_C \hat{B}$. $\square$

Exercises

Exercise A.3. (strong dividing and þ-forking)

Let us say that a formula $\varphi(x, b, d)$ *divides quite strongly* over C if the set $\{\varphi(x, b', d) \mid b' \models \operatorname{tp}(b/C)\}$ of formulas *up to equivalence* is k-inconsistent for some $k < \omega$. Clearly if a formula strongly divides then it also divides quite strongly, though the converse does not hold in general. Show that the converse does hold after passing to þ-forking:

If there is a tuple c such that $\varphi(x, b, d)$ divides quite strongly over Cc, then $\varphi(x, bd)$ is equivalent to a formula that strongly divides over Ccd. (Hence $\varphi(x, bd)$ þ-forks over C.)

Notes

The content of this section is original inasmuch as it proves the equivalence of a new definition of thorn-forking to Alf Onshuus' definition.

It should be noted that there are some ambiguities in [Ons03a, Definition 2.1] that I passed over in silence. Exercise A.3 shows that these do not affect the definition of þ-forking.

A.2 Solutions for exercises

Solution to Exercise 1.5. (relations between axioms, existence and symmetry)

(i) $A \mathop{\smile\!\!\!\!\!\!\;\mid}_C B$ implies $B \mathop{\smile\!\!\!\!\!\!\;\mid}_C A$ by symmetry. Applying extension to $\hat{A} = AC \supseteq A$ we get $B' \equiv_{AC} B$ such that $B' \mathop{\smile\!\!\!\!\!\!\;\mid}_C \hat{A}$, i. e., $B' \mathop{\smile\!\!\!\!\!\!\;\mid}_C AC$. By invariance also $B \mathop{\smile\!\!\!\!\!\!\;\mid}_C \hat{A}$. Hence $AC \mathop{\smile\!\!\!\!\!\!\;\mid}_C B$ by symmetry.

(ii) Let $A_0 \subseteq A$ be any finite subset and note that $A_0 \mathop{\smile\!\!\!\!\!\!\;\mid}_C C$ by local character and base monotonicity. So by finite character, $A \mathop{\smile\!\!\!\!\!\!\;\mid}_C C$ holds. Now for any B we can use extension to find $A' \equiv_C A$ such that $A' \mathop{\smile\!\!\!\!\!\!\;\mid}_C BC$, so $A' \mathop{\smile\!\!\!\!\!\!\;\mid}_C B$ by monotonicity.

(iii) Suppose $A \mathop{\smile\!\!\!\!\!\!\;\mid}_C B$ and $B \subseteq \hat{B}$. By existence there is $A' \equiv_{BC} A$ such that $A' \mathop{\smile\!\!\!\!\!\!\;\mid}_{BC} \hat{B}$. By invariance, $A' \mathop{\smile\!\!\!\!\!\!\;\mid}_C B$. Using Remark 1.3 on the other side, which is possible because of symmetry, we get $A' \mathop{\smile\!\!\!\!\!\!\;\mid}_C \hat{B}$.

Solution to Exercise 1.6. (local character)

(i) It easily follows from existence that $A \mathop{\smile\!\!\!\!\!\!\;\mid}_A B$. Also, $A \mathop{\smile\!\!\!\!\!\!\;\mid}_B B$ by existence and invariance. Therefore we can choose $C_1 = B$ and $C_2 = A$.

(ii) By invariance the statement is clear for finite sets A. Given arbitrary sets A and B, we can find for every finite subset $A_0 \subseteq A$ a subset $C_0 \subseteq B$ such that

$A_0 \mathop{\smile\hspace{-0.9em}|}_{C_0} B$ and $|C_0| < \kappa$. Let C be the union of all these sets C_0. Then $A \mathop{\smile\hspace{-0.9em}|}_{C} B$ by finite character and base monotonicity, and clearly $|C| < \kappa + |T|^+$.

Solution to Exercise 1.7. (modularity and distributivity)

(i) It is sufficient to show: If A, B and $C = \operatorname{acl} C$ are s. t. $A \cap C = B \cap C = \emptyset$, then there is $A' \equiv_C A$ s. t. $A' \cap B = \emptyset$. If this were false, then by compactness there would be a counter-example with A and B finite. Towards a contradiction let A, B and $C = \operatorname{acl} C$ be s. t. $A \cap C = B \cap C = \emptyset$, $A' \cap B \neq \emptyset$ for all $A' \equiv_C A$, B finite and $|A|$ minimal for these properties. Let $A_* \subseteq A$ be a maximal subset of A s. t. $\{A' \mid A' \equiv_C A \text{ and } A_* \subseteq A'\}$ is infinite. By minimality of $|A|$ there is $A'_* \equiv_C A_*$ s. t. $A'_* \cap B = \emptyset$. We may assume $A'_* = A_*$, so $A_* \cap B = \emptyset$. For every $b \in B$, by maximality of A_* there are only finitely many $A' \equiv_C A$ s. t. $A_* \cup b \subseteq A'$. Hence there are only finitely many $A' \equiv_C A$ s. t. $A_* \subseteq A'$ and $A' \cap B \neq \emptyset$. Since $\{A' \mid A' \equiv_C A \text{ and } A_* \subseteq A'\}$ is infinite, there is $A' \equiv_C A$ s. t. $A_* \subseteq A$ and $A' \cap B = \emptyset$, a contradiction.

(ii) Invariance, monotonicity and normality are obvious.

Finite character: Suppose $d \in \operatorname{acl}(AC) \cap \operatorname{acl}(BC) \setminus \operatorname{acl} C$. Then d is already algebraic over a finite tuple $\bar{a}\bar{c}$ with $\bar{a} \in A$ and $\bar{c} \in C$, and also over a finite tuple $\bar{b}\bar{c}'$ with $\bar{b} \in B$ and $\bar{c}' \in C$. Hence $d \in \operatorname{acl}(C\bar{a}) \cap \operatorname{acl}(C\bar{b}) \setminus \operatorname{acl} C$.

Transitivity: Suppose $D \subseteq C \subseteq B$. If $\operatorname{acl} B \cap \operatorname{acl}(AC) \subseteq \operatorname{acl} C$ and $\operatorname{acl} C \cap \operatorname{acl}(AD) \subseteq \operatorname{acl} D$, then $\operatorname{acl} B \cap \operatorname{acl}(AD) \subseteq \operatorname{acl} C \cap \operatorname{acl}(AD) \subseteq \operatorname{acl} D$.

Extension: Using Exercise 1.5 (iii) this follows from (i), symmetry and the other axioms already shown to hold.

Local character: Given sets A and B, construct sets $C_i \subseteq B$ and D_i ($i < \omega$) as follows: $C_0 = D_0 = \emptyset$. $D_{i+1} = \operatorname{acl}(AC_i) \cap \operatorname{acl} B$. For every $d \in D_{i+1}$ let $\bar{c}_d \in B$ be a finite tuple such that $d \in \operatorname{acl} \bar{c}_d$. Let C_{i+1} be the union over all tuples $\bar{c}_d$. Let $C = \bigcup_{i<\omega} C_i$. It is easy to see that $C \subseteq B$ and $|C| \leq |T| + |A|$. Moreover, if $d \in \operatorname{acl}(AC) \cap \operatorname{acl}(BC)$, then already $d \in \operatorname{acl}(AC_i) \cap \operatorname{acl}(BC) \subseteq D_{i+1}$ for some $i < \omega$, hence $d \in \operatorname{acl} C_{i+1} \subseteq \operatorname{acl} C$.

(iii) Let A and $B \supseteq C$ be algebraically closed sets. First note that $B \cap \operatorname{acl}(AC) \supseteq \operatorname{acl}((B \cap A)C)$ holds without any further assumptions. Now suppose $\mathop{\smile\hspace{-0.9em}|}^{\mathrm{a}}$ satisfies the base monotonicity axiom. Then $A \mathop{\smile\hspace{-0.9em}|}^{\mathrm{a}}_{A \cap B} B$ implies $A \mathop{\smile\hspace{-0.9em}|}^{\mathrm{a}}_{(A \cap B)C} B$. Hence $B \cap \operatorname{acl}(AC) \subseteq \operatorname{acl}((B \cap A)C)$.

Conversely, suppose the modular law holds, $A \mathop{\smile\hspace{-0.9em}|}^{\mathrm{a}}_{C} B$ and $C \subseteq C' \subseteq B$. Then $\operatorname{acl} B \cap \operatorname{acl}(AC') \subseteq \operatorname{acl}((\operatorname{acl} B \cap \operatorname{acl} A)C')$ by modularity and $C' \subseteq B$. Note that $\operatorname{acl} B \cap \operatorname{acl} A \subseteq C \subseteq C'$, so $\operatorname{acl} B \cap \operatorname{acl}(AC') \subseteq \operatorname{acl} C'$. Hence $A \mathop{\smile\hspace{-0.9em}|}^{\mathrm{a}}_{C'} B$.

(iv) Let A, B, C be algebraically closed sets. First note that $\operatorname{acl}((A \cap B)C) \subseteq \operatorname{acl}(AC) \cap \operatorname{acl}(BC)$ holds without any further assumptions. Now suppose $\mathop{\smile\hspace{-0.9em}|}^{\mathrm{a}}$ is perfectly trivial. Since $A \mathop{\smile\hspace{-0.9em}|}_{A \cap B} B$ it follows that $A \mathop{\smile\hspace{-0.9em}|}_{(A \cap B)C} B$ as well, hence $AC \mathop{\smile\hspace{-0.9em}|}_{(A \cap B)C} BC$ (by base monotonicity, which can be applied on both sides due to symmetry), hence $\operatorname{acl}(AC) \cap \operatorname{acl}(BC) \subseteq \operatorname{acl}((A \cap B)C)$.

Conversely, suppose the lattice is distributive, $A \mathop{\smile\hspace{-0.9em}|}_{C} B$ and $C' \supseteq C$. Then $\operatorname{acl}(AC') \cap \operatorname{acl}(BC') = \operatorname{acl}((\operatorname{acl} A \cap \operatorname{acl} B)C') \subseteq \operatorname{acl}(CC') = \operatorname{acl} C'$, hence $A \mathop{\smile\hspace{-0.9em}|}_{C'} B$.

Solution to Exercise 1.8. (concerning Example 1.4)

We write $[a, b]$ for the set of nodes on the path from a to b if this path exists. Let $\mathrm{d}(a, b)$ be the size of $[a, b]$ minus one, or ∞.

$A \mathop{\smile\!\!\!\!\!\!|}_C B \implies AC \mathop{\smile\!\!\!\!\!\!|}_C B$ is trivial.

For $A \mathop{\smile\!\!\!\!\!\!|}_C B \implies \operatorname{acl} A \mathop{\smile\!\!\!\!\!\!|}_C B$ suppose $A \mathop{\smile\!\!\!\!\!\!|}_C B$, $a \in \operatorname{acl} A$ and $b \in B$. Let $a_1, a_2 \in A$ be s. t. $a \in [a_1, a_2]$. Suppose $[a, b]$ exists. Let $d \in [a_1, a_2]$ be s. t. $\mathrm{d}(d, b)$ is minimal. Then $[a_1, b] = [a_1, d] \cup [d, b]$ and $[a_2, b] = [a_2, d] \cup [d, b]$. Since $[a_1, b]$ and $[a_2, b]$ meet $\operatorname{acl} C$, it follows that $[d, b]$ meets $\operatorname{acl} C$ (otherwise $d \in \operatorname{acl} C$, a contradiction). Hence $[a, b] = [a, d] \cup [d, b]$ also meets $\operatorname{acl} C$.

Invariance, monotonicity, finite character, base monotonicity, normality and anti-reflexivity are trivial.

Transitivity: Suppose $B \mathop{\smile\!\!\!\!\!\!|}_C A$, $C \mathop{\smile\!\!\!\!\!\!|}_D A$ and $D \subseteq C \subseteq B$. We need to show that $B \mathop{\smile\!\!\!\!\!\!|}_D A$, i. e., every path from B to A meets $\operatorname{acl} D$. Let $b \in B$, $a \in A$ be s. t. $[b, a]$ exists. Let $d \in [b, a] \cap \operatorname{acl} C$. Then $[d, a] \subseteq [b, a]$, and $[d, a]$ meets $\operatorname{acl} D$ by $\operatorname{acl} C \mathop{\smile\!\!\!\!\!\!|}_D A$. Hence $[b, a]$ meets $\operatorname{acl} D$.

Extension: We prove existence instead: Given A, B, C write A as a disjoint union $A = (A \cap \operatorname{acl} C) \cup \bigcup A_i$, where each A_i is the intersection of A with a connected component of $\mathcal{M} \setminus \operatorname{acl} C$. Let A_i' be s. t. $A_i' \equiv_{\operatorname{acl} C} A_i$, and the connected component of A_i' in $\operatorname{acl} C$ avoids B and the other A_i'. Then $A' = (A \cap \operatorname{acl} C) \cup \bigcup A_i' \equiv_{\operatorname{acl} C} A$, and $A' \mathop{\smile\!\!\!\!\!\!|}_C B$. Since $\mathop{\smile\!\!\!\!\!\!|}$ is obviously symmetric, extension now follows as in Exercise 1.5.

Local character: For $a \in A$ s. t. a path from a to B exists let $c_a \in \operatorname{acl} B$ be s. t. $\mathrm{d}(a, c_a)$ is minimal, and let b_a, $b_{a'} \in B$ be s. t. $c_a \in [b_a, b_{a'}]$. Then $C = \{c_a \mid a \in A\} \subseteq B$, $|C| \leq 2\,|A|$, and every path from $a \in A$ to B meets $\operatorname{acl} C$ in c_a.

Solution to Exercise 1.24. (dividing and forking of formulas)

(i) First suppose $\bar{b} \in BC$, $\models \varphi(\bar{a}, \bar{b})$, and $\varphi(\bar{x}, \bar{b})$ divides over C. Let this be witnessed by $(\bar{b}_i)_{i<\omega}$ such that $\bar{b}_i \equiv_C \bar{b}$ and $\{\varphi(\bar{x}, \bar{b}_i) \mid i < \omega\}$ is k-inconsistent. It is not hard to see that we may assume that $(\bar{b}_i)_{i<\omega}$ is C-indiscernible, and that $\bar{b}_0 = \bar{b} \in BC$. Thus $(\bar{b}_i)_{i<\omega}$ witnesses that $\bar{a} \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_C B$.

For the converse, suppose $\bar{a} \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_C B$. This is witnessed by a sequence $(\bar{b}_i)_{i<\omega}$ of C-indiscernibles with $\bar{b}_0 \in BC$ and such that there is no $\bar{a}' \equiv_{BC} \bar{a}$ such that $(\bar{b}_i)_{i<\omega}$ is $\bar{a}'C$-indiscernible. We may assume that $\bar{b}_0$ enumerates all elements of BC. (This involves extracting a sequence of indiscernibles.) Let $p(\bar{x}; \bar{y}) = \operatorname{tp}(\bar{a}\bar{b}_0/C)$. Then $\bigcup_{i<\omega} p(\bar{x}; \bar{b}_i)$ is inconsistent. (Otherwise the set would still be consistent after extending the sequence $(\bar{b}_i)_{i<\omega}$. We could realise it by $\bar{a}^*$, say, and extract an $\bar{a}^*C$-indiscernible sequence $(\bar{b}_i^*)_{i<\omega}$. The C-automorphism taking $(\bar{b}_i^*)_{i<\omega}$ to $(\bar{b}_i)_{i<\omega}$ would take $\bar{a}^*$ to $\bar{a}' \equiv_{BC} \bar{a}$ because $\bar{a}'\bar{b}_0 \equiv_C \bar{a}^*\bar{b}_0^* \equiv_C \bar{a}\bar{b}_0$.)

(ii) We prove only the harder direction. Suppose $\bar{a} \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_C B$, so there is $\hat{B} \supseteq B$ such that $\bar{a}' \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_C \hat{B}$ for all $\bar{a}' \equiv_{BC} \bar{a}$. By (i), $\operatorname{tp}(\bar{a}/BC) \cup \{\neg\varphi(\bar{x}, \bar{b}) \mid \bar{b} \in \hat{B}C;\ \ \varphi(\bar{x}, \bar{b})$ *divides over* $C\}$ is inconsistent. So there are a formula $\psi(\bar{x}, \bar{b}) \in \operatorname{tp}(\bar{a}/BC)$ and formulas $\varphi_i(\bar{x}, \bar{b}_i)$, $i < k$ dividing over C with parameters $\bar{b}_i \in \hat{B}C$ such that $\psi(\bar{x}, \bar{b})$ implies the disjunction $\bigvee_{i<k} \varphi_i(\bar{x}, \bar{b}_i)$.

(iii) Suppose $\varphi(\bar{x};\bar{b})$ does not divide over C. Let $(\bar{b}_i)_{i<\kappa}$ be a Morley sequence for $\operatorname{tp}(\bar{b}/C)$. Then $\{\varphi(\bar{x};\bar{b}_i) \mid i<\kappa\}$ is consistent, hence realised by a tuple $\bar{a}$, say. If κ is sufficiently big, we can extract from $(\bar{b}_i)_{i<\kappa}$ a $C\bar{a}$-indiscernible sequence $(\bar{b}'_i)_{i<\omega}$. By Proposition 1.12, $\bar{a} \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_{C} \bar{b}'_0$, hence $\bar{a} \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_{C} \bar{b}$. Since $\models \varphi(\bar{a};\bar{b})$ and $\bar{a} \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_{C} \bar{b}$, $\varphi(\bar{x};\bar{b})$ does not fork over C.

Solution to Exercise 1.25.(additional properties of $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{d}}$)

(i) Let $(\bar{a}_i)_{i<\omega}$ be a sequence of B-indiscernibles. For a sufficiently big cardinal κ let $(\bar{a}_i)_{i<\omega}$ be a B-indiscernible extension. (This exists by compactness.) Let $\bar{b}_0$ be an enumeration of $\operatorname{acl} B$. For every $i \in \kappa \setminus \{0\}$ let $\bar{b}_i$ be such that $\bar{a}_0\bar{b}_0 \equiv_B \bar{a}_i\bar{b}_i$. If κ was chosen big enough we can extract from the sequence $(\bar{a}_i\bar{b}_i)_{i<\kappa}$ a B-indiscernible sequence $(\bar{a}'_i\bar{b}'_i)_{i<\omega}$. Now it is easy to see that we could have extended the tuples $\bar{a}_i$ in such a way by tuples $\bar{b}_i$ that $(\bar{a}_i\bar{b}_i)_{i<\omega}$ is B-indiscernible.

We now show $\bar{b}_0 = \bar{b}_1$. Otherwise there is an index j such that $\bar{b}_0^j \neq \bar{b}_1^j$. But then $\bar{b}_i^j \neq \bar{b}_{i'}^j$ for all $i \neq i'$, a contradiction since all $\bar{b}_i^j$ satisfy the same algebraic type over B. Therefore $\bar{b}_0 = \bar{b}_1$, hence $\bar{b}_i = \bar{b}_j$ for all $i, j < \omega$. Now it easily follows that $(\bar{a}_i)_{i<\omega}$ is $\operatorname{acl} B$-indiscernible.

(ii) Suppose $b_0 \in B \setminus \operatorname{acl} C$. Then for every κ there is a C-indiscernible sequence $(b_i)_{i<\omega}$ of distinct realisations of $\operatorname{tp}(b_0/B)$. (Start with a very long sequence of distinct realisations and extract a sequence of indiscernibles from it.) By $A \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{d}}_{C} B$ there is $A' \equiv_{BC} A$ such that $(b_i)_{i<\omega}$ is $A'C$-indiscernible. By (i) the sequence is also $\operatorname{acl}(A'C)$-indiscernible. Hence $b_0 \notin \operatorname{acl}(A'C)$.

(iii) Suppose $\bar{b}_0 \in \operatorname{acl}(BC)$ and $(\bar{b}_i)_{i<\omega}$ is a sequence of C-indiscernibles. First we note that we may assume that $\bar{b}_0$ actually enumerates all of $\operatorname{acl}(BC)$: For a sufficiently big cardinal κ we extend the sequence to a sequence $(\bar{b}_i)_{i<\kappa}$ of C-indiscernibles. Let $\bar{b}'_0 \supseteq \bar{b}_0$ be a tuple enumerating $\operatorname{acl}(BC)$. Then we can find for every $i \in \kappa \setminus \{0\}$ a tuple $\bar{b}'_i \supseteq \bar{b}_i$ s. t. $\bar{b}'_i \equiv_C \bar{b}_i$. Now extract a sequence of C-indiscernibles from $(\bar{b}'_i)_{i<\kappa}$.

Let $\bar{b}_0^* \subseteq \bar{b}_0$ be the subtuple of $\bar{b}_0$ that enumerates BC. Let $\bar{b}_i^*$ be the tuple corresponding to $\bar{b}_0^*$ in $\bar{b}_i$. Then there is $A' \equiv_{BC} A$ such that $(\bar{b}_i^*)$ is $A'C$-indiscernible. By compactness it is possible to extend the sequence to a sequence $(\bar{b}_i)_{i<\kappa}$ that is C-indiscernible and such that $(\bar{b}_i^*)_{i<\kappa}$ is $A'C$-indiscernible if $\bar{b}_i^*$ is defined as before for $i \geq \omega$. If κ is big enough we can extract a sequence of $A'C$-indiscernibles. Thus we get a sequence $\bar{b}'_i$ of $A'C$-indiscernibles such that $\bar{b}'_0 \equiv_{A'} \bar{b}_0$ and $\bar{b}'^*_0 \equiv_{A'C} \bar{b}^*_0$. Since $\bar{b}'_0$ enumerates $\operatorname{acl}(BC)$ and $\bar{b}'^*_0$ enumerates BC it follows that there is a permutation of $\bar{b}'_0$ that fixes $\bar{b}'^*_0$ and such that $\bar{b}'_0 \equiv_{A'C} \bar{b}_0$. Now we can take an $A'C$-automorphism of the big model that maps $\bar{b}'_0$ to $\bar{b}_0$ and fixes the permutation issue. It takes $(\bar{b}'_i)_{i<\omega}$ to an $A'C$-indiscernible sequence $(\bar{b}''_i)_{i<\omega}$ such that $\bar{b}''_0 = \bar{b}_0$.

(iv) Suppose $A \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{d}}_{C} B$. First note that $A \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{d}}_{C} \operatorname{acl}(BC)$ by (iii). Now suppose $C \subseteq C' \subseteq \operatorname{acl}(BC)$. Since $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{d}}$ satisfies base monotonicity by Lemma 1.22, we have $A \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{d}}_{C'} \operatorname{acl}(BC)$. Hence $\operatorname{acl}(AC') \cap \operatorname{acl}(BC) = \operatorname{acl} C$ by (ii).

Solution to Exercise 1.34. (concerning Example 1.31)

We show that $A \not\mathop{\smile\!\!\!\!\!\mid}_{C} B \implies A \not\mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}_{C} B$, so suppose $A \not\mathop{\smile\!\!\!\!\!\mid}_{C} B$. Then there are nodes $a \in A$ and $b \in B$ such that $[a, b]$ (exists and) avoids $\operatorname{acl} C$. First consider

the case that there is a node $c \in C$ in the same connected component as a and b. Choose $c' \in [a,b]$ s. t. $\mathrm{d}(c',c)$ is minimal. Then $c' \in \operatorname{acl}(aC) \cap \operatorname{acl}(bC) \setminus \operatorname{acl} C$, so $a \mathop{\not\perp\!\!\!\perp}^{\mathrm{M}}_C b$ and hence $A \mathop{\not\perp\!\!\!\perp}^{\text{þ}}_C B$. In the second case C does not meet the connected component of a and b. Let d, e be two distinct neighbours of b. If, towards a contradiction, $A \mathop{\perp\!\!\!\perp}^{\text{þ}}_C B$, then also $a \mathop{\perp\!\!\!\perp}^{\text{þ}}_C b$, hence there must be $d'e' \equiv_{bC} de$ such that $a \mathop{\perp\!\!\!\perp}^{\mathrm{M}}_C bd'e'$. Since d', e' are distinct neighbours of b, one of them, d' say, is not in $[a,b]$. But then $b \in [a,d']$, so $b \in \operatorname{acl}(aCd') \cap \operatorname{acl}(bCd') \setminus \operatorname{acl}(Cd')$, hence $a \mathop{\not\perp\!\!\!\perp}^{\mathrm{M}}_C bd'$, a contradiction. Therefore $A \mathop{\not\perp\!\!\!\perp}^{\text{þ}}_C B$ in both cases.

Solution to Exercise 1.35. (concerning Example 1.32)

Since $\operatorname{acl} A = A$ for all A, the lattice of algebraically closed sets is distributive. Hence $\mathop{\perp\!\!\!\perp}^{\mathrm{a}}$ is a (perfectly trivial) strict independence relation by Exercise 1.7. $\mathop{\perp\!\!\!\perp}^{\mathrm{a}}$ is clearly coarser than $\mathop{\perp\!\!\!\perp}^{\text{þ}}$. On the other hand $\mathop{\perp\!\!\!\perp}^{\text{þ}}$ is coarser than $\mathop{\perp\!\!\!\perp}^{\mathrm{a}}$ by Theorem 1.30. Therefore $\mathop{\perp\!\!\!\perp}^{\mathrm{a}} = \mathop{\perp\!\!\!\perp}^{\text{þ}}$. Clearly $A \mathop{\perp\!\!\!\perp}^{\mathrm{a}}_C B$ iff $AC \cap BC = C$, iff $A \cap B \subseteq C$.

Solution to Exercise 1.36. (concerning Example 1.33)

For thorn-forking for T we are in the situation of Example 1.32. In T^{eq} we have $\operatorname{acl}^{\mathrm{eq}}(AB) = \operatorname{acl}^{\mathrm{eq}} A \cup \operatorname{acl}^{\mathrm{eq}} B$, so the lattice of algebraically closed sets for T^{eq} is also distributive and we can argue exactly as in Example 1.32.

Solution to Exercise 1.56. (concerning Example 1.48)

Transitivity: Suppose $B \mathop{\perp\!\!\!\perp}'_{CD} A$ and $C \mathop{\perp\!\!\!\perp}'_D A$. Consider the three cases for $C \mathop{\perp\!\!\!\perp}'_D A$. If $A \subseteq D$, then $BC \mathop{\perp\!\!\!\perp}'_D A$ follows trivially. If $C \subseteq D$, then $CD = D$, so $BC \mathop{\perp\!\!\!\perp}'_D A$ also follows trivially. Otherwise D is infinite, $B \mathop{\perp\!\!\!\perp}_{CD} A$ and $C \mathop{\perp\!\!\!\perp}_D A$, hence $BC \mathop{\perp\!\!\!\perp}_D A$, hence $BC \mathop{\perp\!\!\!\perp}'_D A$. Local character: Let κ be suitable for local character of $\mathop{\perp\!\!\!\perp}$. For B and finite A there is $C \subseteq B$ s. t. $|C| < \kappa$ and $A \mathop{\perp\!\!\!\perp}_C B$. If B is finite choose $C' = B$, otherwise choose C' s. t. $C \subseteq C' \subseteq B$ and $\aleph_0 \leq |C'| < \kappa$. No extension, no existence: Consider the case of finite C.

Solution to Exercise 2.6. (strong finite character of $\mathop{\perp\!\!\!\perp}^{\mathrm{M}}$)

Suppose $A \mathop{\not\perp\!\!\!\perp}^{\mathrm{M}}_C B$. Then there is D such that $C \subseteq D \subseteq \operatorname{acl}(BC)$ and an element $e \in \big(\operatorname{acl}(AD) \cap \operatorname{acl}(BD)\big) \setminus \operatorname{acl} D$. Let $\bar{a}$, $\bar{b}$ and $\bar{c}$ be enumerations of A, B and C, respectively.

Since $e \in \operatorname{acl}(AD)$, we can find a finite tuple $\bar{d} \in D$ and an algebraic formula $\alpha(u, \bar{a}, \bar{d})$ such that $\models \alpha(d, \bar{a}, \bar{d})$. Then for appropriate $k < \omega$, e satisfies the formula $\alpha'(u, \bar{a}, \bar{d})$ defined as $\alpha(u, \bar{a}, \bar{d}) \wedge \exists_{\leq k} u' \alpha(u', \bar{a}, \bar{d})$.

Since $e \in \operatorname{acl}(BD) = \operatorname{acl}(BC)$, there is an algebraic formula $\beta(u, \bar{b}, \bar{c})$ such that $\models \beta(e, \bar{b}, \bar{c})$. Let $e_0, \ldots, e_{n-1}$ be all the realisations of $\beta(u, \bar{b}, \bar{c})$ that are in $\operatorname{acl} D$.

Let $\chi(u, \bar{d}^*)$ be an algebraic formula with parameters in D that is satisfied at least by $e_0, \ldots, e_{n-1}$. We may assume that $\bar{d} = \bar{d}^*$. Note that every element e' that satisfies $\beta(u, \bar{b}, \bar{c})$, either satisfies $\chi(u, \bar{d})$ or is not algebraic over $C\bar{d}$ at all.

Let $\delta(\bar{v}, \bar{b}, \bar{c})$ be an isolating formula in the algebraic type $\operatorname{tp}(\bar{d}/B \cup C)$. Note that for any $\bar{d}'$ satisfying $\delta(\bar{v}, \bar{b}, \bar{c})$, every element e' that satisfies $\beta(u, \bar{b}, \bar{c})$ either satisfies $\chi(u, \bar{d}')$ or is not algebraic over $C\bar{d}'$ at all.

Let $\varphi(\bar{x}, \bar{b}, \bar{c})$ be the formula defined as

$$\exists u \exists \bar{v} \big(\delta(\bar{v}, \bar{b}, \bar{c}) \wedge \alpha'(u, \bar{x}, \bar{v}) \wedge \beta(u, \bar{b}, \bar{c}) \wedge \neg\chi(u, \bar{v})\big).$$

$\varphi(\bar{x}, \bar{b}, \bar{c})$ has the property desired:

First note that e and $\bar{d}$ witness that $\models \varphi(\bar{a}, \bar{b}, \bar{c})$ holds. On the other hand, suppose $\models \varphi(\bar{a}', \bar{b}, \bar{c})$ holds and let e' and $\bar{d}'$ witness this, i.e.,

$$\models \delta(\bar{d}', \bar{b}, \bar{c}) \wedge \alpha'(e', \bar{a}', \bar{d}') \wedge \beta(e', \bar{b}, \bar{c}) \wedge \neg\chi(e', \bar{d}').$$

Let $D' = C\bar{d}'$. From $\delta(\bar{d}', \bar{b}, \bar{c})$ we get $C \subseteq D' \subseteq \operatorname{acl}(BC)$. From $\models \alpha'(e', \bar{a}', \bar{d}')$ we get $e' \in \operatorname{acl}(\bar{a}'\bar{d}') \subseteq \operatorname{acl}(D'\bar{a}')$. From $\models \beta(e', \bar{b}, \bar{c}) \wedge \neg\chi(e', \bar{d}')$ we get $e' \in \operatorname{acl}(BC)$ and $e' \notin \operatorname{acl}(C\bar{d}') = \operatorname{acl} D'$. Hence e' witnesses $\operatorname{acl}(D'\bar{a}) \cap \operatorname{acl}(BD') \supsetneq \operatorname{acl} D'$.

Solution to Exercise 2.7. (Figure 2.1)

We need to prove that the relations defined in Examples 1.47, 1.48, 1.50 and 1.55 satisfy strong finite character. For Example 1.47 this is trivial. In Examples 1.50 and 1.55 we are dealing with $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}}$, so we can just use Exercise 2.6.

Example 1.48: We can choose for $\mathop{\smile\!\!\!\!\!\!|}$ a strict independence relation satisfying strong finite character. For infinite C we have $A \mathop{\smile\!\!\!\!\!\!|}'_C B \iff A \mathop{\smile\!\!\!\!\!\!|}_C B$, so we can just use strong finite character of $\mathop{\smile\!\!\!\!\!\!|}$. For finite C, suppose $A \not\mathop{\smile\!\!\!\!\!\!|}'_C B$. Choose $a \in A \setminus C$, $b \in B \setminus C$, and let $\bar{c}$ be an enumeration of C. Let $\varphi(\bar{a}, \bar{b}, \bar{c})$ express that neither a nor b is among the elements of the tuple $\bar{c}$.

Solution to Exercise 2.8. (alternative definition for strong finite character)

For one direction, suppose $\mathop{\smile\!\!\!\!\!\!|}$ satisfies extension, and for any sequence of variables $\bar{x}$ and any sets B and C, the set $\{\operatorname{tp}(\bar{a}/BC) \mid \bar{a} \not\mathop{\smile\!\!\!\!\!\!|}_C B\}$ is an open subset of $\mathrm{S}_{\bar{x}}(BC)$. We will show that $\mathop{\smile\!\!\!\!\!\!|}$ satisfies strong finite character. So suppose $A \not\mathop{\smile\!\!\!\!\!\!|}_C B$. Let $\bar{a}$ be an enumeration of A. The basic open sets of $\mathrm{S}_{\bar{x}}(BC)$ are those of the form $\{\operatorname{tp}(\bar{a}/BC) \mid \models \varphi(\bar{x}, \bar{b}, \bar{c})\}$, where $\varphi(\bar{x}, \bar{y}, \bar{z})$ is a formula without parameters and $\bar{b} \in B$ and $\bar{c} \in C$ are finite tuples. Let $\varphi(\bar{x}, \bar{y}, \bar{z})$ and $\bar{b} \in B$, $\bar{c} \in C$ be such that $\operatorname{tp}(\bar{a}/BC) \in \{\operatorname{tp}(\bar{a}/BC) \mid \models \varphi(\bar{a}, \bar{b}, \bar{c})\} \subseteq \{\operatorname{tp}(\bar{a}/BC) \mid \bar{a} \not\mathop{\smile\!\!\!\!\!\!|}_C B\}$. Then clearly $\models \varphi(\bar{a}, \bar{b}, \bar{c})$ and $\bar{a}' \not\mathop{\smile\!\!\!\!\!\!|}_C B$ for all $\bar{a}'$ satisfying $\models \varphi(\bar{a}', \bar{b}, \bar{c})$. By extension, $\bar{a}' \not\mathop{\smile\!\!\!\!\!\!|}_C \bar{b}$ for all $\bar{a}'$ satisfying $\models \varphi(\bar{a}', \bar{b}, \bar{c})$. Since only finitely many variables from $\bar{x}$ really occur in $\varphi(\bar{x}, \bar{y}, \bar{z})$, we may replace $\bar{x}$ by a finite subtuple $\bar{x}_0$ and $\bar{a}$ by a finite subtuple $\bar{a}_0$.

Conversely, suppose $\mathop{\smile\!\!\!\!\!\!|}$ satisfies monotonicity and strong finite character. We will show that for any sequence of variables $\bar{x}$ and any sets B and C, the set $\{\operatorname{tp}(\bar{a}/BC) \mid \bar{a} \not\mathop{\smile\!\!\!\!\!\!|}_C B\}$ is an open subset of $\mathrm{S}_{\bar{x}}(BC)$. So suppose $\operatorname{tp}(\bar{a}/BC) \in \{\operatorname{tp}(\bar{a}/BC) \mid \bar{a} \not\mathop{\smile\!\!\!\!\!\!|}_C B\}$. Let A be the set of elements of $\bar{a}$. By monotonicity and strong finite character there are tuples $\bar{e} \in A$, $\bar{b} \in B$, $\bar{c} \in C$ and a formula $\varphi(\bar{u}, \bar{y}, \bar{z})$ without parameters such that $\models \varphi(\bar{e}, \bar{b}, \bar{c})$, and $\bar{e}' \not\mathop{\smile\!\!\!\!\!\!|}_C B$ for all $\bar{e}'$ satisfying $\models \varphi(\bar{e}', \bar{b}, \bar{c})$. We can write $\bar{e} = (a_{i_0}, a_{i_1}, \ldots, a_{i_{k-1}})$. Let $\psi(\bar{x}) \equiv \varphi(x_{i_0}, x_{i_1}, \ldots, x_{i_{k-1}})$. Then $\models \psi(\bar{a}, \bar{b}, \bar{c})$, and $\bar{a}' \not\mathop{\smile\!\!\!\!\!\!|}_C B$ for all $\bar{a}'$ such that $\models \psi(\bar{a}', \bar{b}, \bar{c})$. So $\operatorname{tp}(\bar{a}/BC) \in \{\operatorname{tp}(\bar{a}'/BC) \mid \models \varphi(\bar{a}', \bar{b}, \bar{c})\} \subseteq \{\operatorname{tp}(\bar{a}'/BC) \mid \bar{a}' \not\mathop{\smile\!\!\!\!\!\!|}_C B\}$.

Solution to Exercise 2.17. (Δ-forking)

First suppose $\operatorname{tp}(\bar{a}/BC)$ Δ-forks over C for some $\Delta \subseteq \Omega$. Then $\operatorname{tp}(\bar{a}/BC)$ implies some disjunction $\bigvee_{i<n} \varphi^i(\bar{x}; \bar{b}^i)$, where each of the formulas $\varphi(\bar{x}; \bar{b}^i)$ (φ^i, ψ^i)-divides over C and $(\varphi^i, \psi^i) \in \Delta \subseteq \Xi$. (Note that we admit $\bar{b}^i \notin BC$.) Let

$\hat{B} = BC\bar{b}^{<n}$. Then every $\bar{a}' \equiv_{BC} \bar{a}$ realises $\operatorname{tp}(\bar{a}/BC)$, so $\models \bigvee_{i<n} \varphi^i(\bar{a}';\bar{b}^i)$, and so $\bar{a}' \mathop{\not\perp\!\!\!\perp}^{\Omega}_{C} \bar{b}^i$ for some $i < n$. Hence $\bar{a}' \mathop{\not\perp\!\!\!\perp}^{\Omega}_{C} \hat{B}$. Therefore $\bar{a} \mathop{\not\perp\!\!\!\perp}^{\Omega*}_{C} \hat{B}$.

Conversely, suppose $\operatorname{tp}(\bar{a}/BC)$ does not Δ-fork over C for any finite $\Delta \subseteq \Omega$. Then for any $\hat{B} \supseteq B$ the partial type

$$\operatorname{tp}(\bar{a}/BC) \cup \big\{\neg\varphi(\bar{x};\bar{b}) \;\big|\; \bar{b} \in \hat{B},\ (\varphi,\psi) \in \Omega{\restriction}\bar{x},\ \varphi(\bar{x};\bar{b})\ \ (\varphi,\psi)\text{-divides over } C\big\}$$

is consistent. Let $\bar{a}'$ realise this type. Then clearly $\bar{a}' \equiv_{BC} \bar{a}$ and $\bar{a}' \mathop{\perp\!\!\!\perp}^{\Omega}_{C} \hat{B}$.

Solution to Exercise 2.18. (more on dividing and forking of formulas)

(i) First suppose $\varphi(\bar{x};\bar{b})$ divides over C, so there is a number $k < \omega$ and a sequence $(\bar{b}_i)_{i<\omega}$ such that $\bar{b}_i \equiv_C \bar{b}$ and $\{\varphi(\bar{x};\bar{b}_i) \mid i < \omega\}$ is k-inconsistent. Consider the formula $\psi(\bar{x}_{<k}) \equiv \neg\exists\bar{x} \bigwedge_{i<k} \varphi(\bar{x};\bar{y}_i)$, which is clearly a k-inconsistency witness for $\varphi(\bar{x};\bar{y})$. Now each $\bar{b}_i$ realises $\operatorname{tp}(\bar{b}/C)$, and $\psi(\bar{b}_{i_0},\ldots,\bar{b}_{i_{k-1}})$ holds for all $i_0 < \ldots < i_{k-1} < \omega$ by k-inconsistency. Therefore $\varphi(\bar{x};\bar{b})$ (φ,ψ)-divides over C. Conversely, suppose that $\varphi(\bar{x};\bar{b})$ (φ,ψ)-divides over C, where $\psi(\bar{y}_{i<k})$ is any inconsistency witness for $\varphi(\bar{x};\bar{y})$. Then there is a sequence $(\bar{b}_i)_{i<\omega}$ such that $\bar{b}_i \equiv_C B$ for all $i < \omega$ and $\psi(\bar{b}_{i_0},\ldots,\bar{b}_{i_{k-1}})$ holds for all $i_0 < \ldots < i_{k-1} < \omega$. Since $\psi(\bar{y}_{i<k})$ is an inconsistency witness for $\varphi(\bar{x};\bar{y})$ this implies that $\bigwedge_{i<k} \varphi(\bar{x};\bar{y}_i)$ is inconsistent for all $i_0 < \ldots < i_{k-1} < \omega$. In other words: $\{\varphi(\bar{x};\bar{b}_i) \mid i < \omega\}$ is k-inconsistent. Therefore $\varphi(\bar{x};\bar{b})$ divides over C.

(ii) $\varphi(\bar{x};\bar{b})$ forks over C iff $\varphi(\bar{x};\bar{b})$ implies a finite disjunction $\bigvee_{i<n} \varphi^i(\bar{x};\bar{y}^i)$ of formulas $\varphi^i(\bar{x};\bar{y}^i)$ each of which divides over C, or, which is equivalent by Exercise 2.18, each of which (φ^i,ψ^i)-divides over C for some inconsistency witness ψ^i for φ^i.

Solution to Exercise 2.19. (concerning Example 2.13)

We prove that $A \mathop{\not\perp\!\!\!\perp}^{\Omega}_{C} B$ iff the condition $|(A \cap B) \setminus C| \geq 2$ holds. $A \mathop{\not\perp\!\!\!\perp}^{\Omega}_{C} B$ iff there are $a, a' \in A$ and $b, b' \in B$ s. t. $\models \varphi(aa', bb')$ and a sequence $(b_i b_i')_{i<\omega}$ s. t. $b_i b_i' \equiv_C bb'$ and $\models \psi(b_i b_i', b_j b_j')$ for all $i < j < \omega$. This is the case iff there are $a, a' \in A \cap BC$ s. t. $a \neq a'$ and a sequence $(b_i b_i')_{i<\omega}$ s. t. $b_i \neq b_j$, $b_j \neq b_j'$ and $b_i b_i' \equiv_C aa'$ for all $i < j < \omega$. This is equivalent to existence of $a, a' \in A \cap BC$ s. t. $a \neq a'$, $a \notin C$ and $a' \notin C$. But that just means $|(A \cap BC) \setminus C| \geq 2$.

Solution to Exercise 2.22. (tree property)

First note that given any formula $\varphi(\bar{x};\bar{y})$ and $k < \omega$, the formula $\psi(\bar{y}_{<k}) \equiv \neg\exists\bar{x} \bigwedge_{i<k} \varphi(\bar{x};\bar{y}_i)$ is the most general k-inconsistency witness for φ in the sense that whenever $\psi'(\bar{y}_{<k})$ is a k-inconsistency witness for φ and $(\bar{b}_i)_{i<\omega}$ is a sequence such that we have $\models \varphi'(\bar{b}_{i_0},\ldots,\bar{b}_{i_{k-1}})$ for all $i_0 < \cdots < i_{k-1} < \omega$, then also $\models \varphi(\bar{b}_{i_0},\ldots,\bar{b}_{i_{k-1}})$ for all $i_0 < \cdots < i_{k-1} < \omega$. Now note that $\mathrm{D}_{\varphi,\psi}(\emptyset) = \infty$ iff the unique element $\xi \in \{(\varphi,\psi)\}^{\omega}$ is a dividing pattern, iff the type $\operatorname{divpat}^{\xi}_{\emptyset}$ is consistent. But the tree described in the exercise is just a partial realisation of $\operatorname{divpat}^{\xi}_{\emptyset}$.

Solution to Exercise 2.30. (symmetry of $\mathop{\perp\!\!\!\perp}^{\Omega}$)

Symmetry of $\mathop{\perp\!\!\!\perp}^{\Omega}$ implies condition (5) of Theorem 2.29, so $\mathop{\perp\!\!\!\perp}^{\Omega*}$ is an independence relation. Concerning $\mathop{\perp\!\!\!\perp}^{\Omega}$, it now follows that $\mathop{\perp\!\!\!\perp}^{\Omega}$ satisfies existence. Using this we can show that $\mathop{\perp\!\!\!\perp}^{\Omega}$ satisfies extension (and hence $\mathop{\perp\!\!\!\perp}^{\Omega} = \mathop{\perp\!\!\!\perp}^{\Omega*}$, and therefore $\mathop{\perp\!\!\!\perp}^{\Omega}$

is also an independence relation): Suppose $A \mathop{\smile\!\!\!\!\!\!|}^{\Omega}_C B$ and $\hat{B} \supseteq B$. Let $\hat{B}' \equiv_{BC} \hat{B}$ be such that $\hat{B}' \mathop{\smile\!\!\!\!\!\!|}^{\Omega}_{BC} A$. Since $B \mathop{\smile\!\!\!\!\!\!|}^{\Omega}_C A$ we can apply transitivity, to get $\hat{B}' \mathop{\smile\!\!\!\!\!\!|}^{\Omega}_C A$ and hence $A \mathop{\smile\!\!\!\!\!\!|}^{\Omega}_C \hat{B}'$.

Solution to Exercise 2.41. (M-symmetry)

(i) Note that $\operatorname{acl}(AC) \cap B \supseteq \operatorname{acl}(C(A \cap B))$ always holds for $C \subseteq B$. Now first suppose $\mathrm{M}(A, B)$ holds and C is a set s. t. $A \cap B \subseteq C \subseteq B$. Then $\operatorname{acl}(AC) \cap \operatorname{acl}(BC) = \operatorname{acl}(A \operatorname{acl} C) \cap \operatorname{acl} B = \operatorname{acl}(\operatorname{acl} C(A \cap B)) = \operatorname{acl} C$. Conversely, suppose $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}}_{A\cap B} B$ and $C \subseteq B$. Let $C' = C(A \cap B)$. Then $\operatorname{acl}(AC) \cap B = \operatorname{acl}(AC') \cap (BC') = \operatorname{acl} C' = \operatorname{acl}(C(A \cap B))$.

(ii) This follows from (i) since $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}}_C B$ holds iff $\operatorname{acl}(AC) \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}}_C \operatorname{acl}(BC)$, iff $\operatorname{acl}(AC) \cap \operatorname{acl}(BC) = \operatorname{acl} C$ and $\mathrm{M}(\operatorname{acl}(AC), \operatorname{acl}(BC))$.

Solution to Exercise 3.5. (independence in a reduct)

(i) Let $\bar{a}$ enumerate A. Let $(\bar{a}_i)_{i<\omega}$ be a $\mathop{\smile\!\!\!\!\!\!|}$-Morley sequence for $\operatorname{tp}(\bar{a}/BC)$. Note that $\operatorname{acl}(C\bar{a}_{<k}) \cap \operatorname{acl}(C\bar{a}_{\geq k}) = \operatorname{acl} C = C$ (for $k < \omega$) by anti-reflexivity of $\mathop{\smile\!\!\!\!\!\!|}$. Since $C \subseteq \operatorname{acl}'(C\bar{a}_{<k}) \cap \operatorname{acl}'(C\bar{a}_{\geq k}) \subseteq \operatorname{acl}(C\bar{a}_{<k}) \cap \operatorname{acl}(C\bar{a}_{\geq k}) = C$ we have $\operatorname{acl}'(C\bar{a}_{<k}) \cap \operatorname{acl}'(C\bar{a}_{\geq k}) = C$. Applying Lemma 3.2 we get $\bar{a} \mathop{\smile\!\!\!\!\!\!|}'_C B$. ($\operatorname{acl}'$ denotes algebraic closure computed in T'.)

(ii) $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ is a strict independence relation for T^{eq} and for T'^{eq}. By elimination of hyperimaginaries and Corollary 3.4 $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ is in fact a canonical independence relation for T'^{eq}. Therefore we can apply (i).

Solution to Exercise 3.21. (weak canonical bases)

(i) For $C \subseteq \operatorname{acl} B$, enumerated as $\bar{c}$, say, there is $\bar{c}'$ such that $\bar{a}\bar{c} \equiv_B \bar{a}'\bar{c}'$. Since $\bar{c} \equiv_C \bar{c}'$, $\bar{c}'$ also enumerates C. Hence $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_C B \iff \bar{a}' \mathop{\smile\!\!\!\!\!\!|}_C B$ by invariance. Both $\operatorname{wcb}(\bar{a}/\operatorname{acl} B)$ and $\operatorname{wcb}(\bar{a}'/\operatorname{acl} B)$ are defined as the smallest such C, so they agree.

(ii) The smallest algebraically closed set $C \subseteq \operatorname{acl} B$ satisfying $\bar{a}' \mathop{\smile\!\!\!\!\!\!|}_C B$ clearly satisfies $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_C B$, so $\operatorname{wcb}(\bar{a}/B) \subseteq C$.

(iii) (3) implies (2) because $\operatorname{wcb}(\bar{a}/C) \subseteq \operatorname{acl} C$.

(2) implies (1): By $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_{\operatorname{wcb}(\bar{a}/BC)} \operatorname{acl}(BC)$, $\operatorname{wcb}(\bar{a}/BC) \subseteq \operatorname{acl} C \subseteq \operatorname{acl}(BC)$ and base monotonicity we have $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_{\operatorname{acl} C} \operatorname{acl}(BC)$, hence $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_C B$.

(1) implies (3): Suppose $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_C B$. For algebraically closed $D \subseteq \operatorname{acl}(BC)$ we have the equivalence $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_D BC \iff \bar{a} \mathop{\smile\!\!\!\!\!\!|}_D C$.

Solution to Exercise 3.29. (1-based theories)

(i) Recall from Exercise 1.7 that the lattice is modular iff $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}}$ is an independence relation. Now suppose T is 1-based. Then $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}}_C B$ implies $\operatorname{acl}(AC) \cap \operatorname{acl}(BC) = \operatorname{acl} C$, so $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_C B$. Since $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ is a strict independence relation, $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_C B$ also implies $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}}_C B$, so $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}} = \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ is an independence relation. Conversely, suppose $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}}$ is an independence relation. Since $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}}$ is strict and $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ is the coarsest strict independence relation because it is canonical, $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}}_{\operatorname{acl} A \cap \operatorname{acl} B}$ implies $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_{\operatorname{acl} A \cap \operatorname{acl} B}$, so T is 1-based.

(ii) If T is 1-based and perfectly trivial, then $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}} = \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$, so the lattice is distributive by Exercise 1.7. Conversely, if the lattice is distributive, then it is also modular, hence T is 1-based and $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}} = \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$. And $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}}$ is perfectly trivial, again by Exercise 1.7.

Solution to Exercise A.3. (strong dividing and þ-forking)

Suppose $\varphi(x, b, d)$ divides quite strongly over Cc. Consider the definable equivalence relation $\epsilon(yu, y'u') \equiv \forall x(\varphi(x, y, u) \leftrightarrow \varphi(x, y', u'))$ and the formula $\psi(x, v) \equiv \exists y \exists u(v = (yu/\epsilon) \wedge \varphi(x, y, u))$. Then $\varphi(x, bd)$ is equivalent to $\psi(x, e)$ where $e = bd/\epsilon$. It is sufficient to show that $\psi(x, e)$ strongly divides over Ccd.

By assumption the set $\{\varphi(x, b', d) \mid b' \models \mathrm{tp}(b/Cc)\}$ of formulas up to equivalence is k-inconsistent for some k. Hence the same is true for the smaller set $\{\varphi(x, b', d) \mid b' \models \mathrm{tp}(b/Ccd)\}$ of formulas up to equivalence which we can also write as $\{\varphi(x, b'd') \mid b'd' \models \mathrm{tp}(bd/Ccd)\}$. Hence the set $\{\psi(x, e') \mid e' \models \mathrm{tp}(e/Ccd)\}$ of formulas up to equivalence is also k-inconsistent. But for this last set the qualification 'up to equivalence' is unnecessary because every e' is a canonical parameter. Therefore $\psi(x, e)$ strongly divides over Ccd.

A.3 Open questions

Question A.1. *If* $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ *satisfies existence, does it follow that* $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}} = \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}$*?*

Question A.2. *If* $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}} = \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}$*, does it follow that* $\mathop{\smile\!\!\!\!\!\!|}^{\text{þ}} = \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}}$*?*

Question A.3. *Does every independence relation have strong finite character?*

Question A.4. *Is every independence relation of the form* $\mathop{\smile\!\!\!\!\!\!|}^{\Omega *}$*?*

I would have liked to provide counterexamples for these four questions, but I did not find any.

Question A.5. *Is there a theory with more than one* strict *independence relation on* T^{eq}*?*

It would be a good thing if not: Given any simple theory, Shelah-forking independence $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ is a strict independence relation on T^{eq}. It follows from Theorem 1.30 that thorn-forking independence is also a strict independence relation on T^{eq}. If $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ and $\mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}$ agree, then we are very close to elimination of hyperimaginaries.

Question A.6. *Is every rosy theory the reduct of a theory with elimination of hyperimaginaries and such that* $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}}$ *is symmetric?*

This is perhaps not important, but it is the second half of a question I have been asking myself privately for years. The first half ('Does symmetry of $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}}$ imply that $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}}$ is an independence relation?') was answered by Theorem 2.39.

Question A.7. *Does 'hyperimaginary thorn-forking' make sense? In other words: How much of Section 1.5 can be carried out for T^{heq} with bounded closure* bdd *instead of algebraic closure* acl*?*

In a simple theory, Shelah-forking can be extended to and has canonical bases in T^{heq}. Therefore Shelah-forking agrees with hyperimaginary thorn-forking in this case. This suggests that perhaps hyperimaginary thorn-forking is a more natural notion than thorn-forking. On the other hand it is not the right notion of independence in o-minimal theories. If hyperimaginary thorn-forking does not make sense in general, then this might be considered a heuristic argument for elimination of hyperimaginaries (at least in a weak sense) in all simple theories. I think Clifton Ealy may have some partial results.

Question A.8. *For T^{heq} define $A \mathop{\smile\!\!\!\!\!\!|}^{\flat}_{C} B \iff \mathrm{bdd}(AC) \cap \mathrm{bdd}(BC) = \mathrm{bdd}\, C$. Does $\mathop{\smile\!\!\!\!\!\!|}^{\flat}$ always satisfy existence (at least over sufficiently saturated models)? Suppose T is such that $\mathop{\smile\!\!\!\!\!\!|}^{\flat}$ satisfies existence and base monotonicity. Does it follow that $\mathop{\smile\!\!\!\!\!\!|}^{\flat}$ satisfies finite character?*

The previous question may be too hard. This is a simpler test question.

An o-minimal theory without weak canonical bases

Up to a very late draft of this thesis I asked if $\mathop{\smile\!\!\!\!\!\!|}^{\flat}$ is canonical for T^{eq} for every o-minimal theory. All o-minimal theories are rosy, and I did not even know a rosy theory such that $\mathop{\smile\!\!\!\!\!\!|}^{\flat}$ is not canonical for T^{eq}. I am grateful to Anand Pillay for directing me to [LP93], which contains a counterexample (Example 4.5) that I present here in a slightly untangled form.

We will construct two o-minimal theories T and T' with elimination of imaginaries such that T' is a reduct of T, $\mathop{\smile\!\!\!\!\!\!|}^{\flat}$ is a canonical independence relation for T, but $\mathop{\smile\!\!\!\!\!\!|}^{\flat}$ for T' does not satisfy the intersection property. T will be interpretable in the theory of $(\mathbb{R}; +, <)$.

Let $\mathbb{R}_<$ and $\mathbb{R}_>$ be two copies of the set $\mathbb{R}$ of real numbers. Consider the following structure $\mathcal{R}$: The domain of $\mathcal{R}$ consists of the disjoint union of $\mathbb{R}_<$, $\mathbb{R}_>$ and a new point $*$. There are constants for the two points -1 and 1 in $\mathbb{R}_<$ as well as for the points -1 and 1 in $\mathbb{R}_>$ and the point $*$. The structure has a linear order $<$ extending the usual order on $\mathbb{R}_<$ and $\mathbb{R}_>$ and such that $\mathbb{R}_< < * < \mathbb{R}_>$. There is also a partial binary function $+$ that is defined on pairs from $\mathbb{R}_<$ and on pairs from $\mathbb{R}_>$ and is interpreted as addition in $\mathbb{R}_<$ or $\mathbb{R}_>$, respectively. Finally, there is a partial function f defined on $\mathbb{R}_<$: the obvious order-preserving isomorphism $\mathbb{R}_< \to \mathbb{R}_>$. (The only purpose of the point $*$ is to make the definable sets $\mathbb{R}_<$ and $\mathbb{R}_>$ definable open intervals. Otherwise $\mathcal{R}$ would not be o-minimal.)

Let T be the theory of $\mathcal{R}$. Since T is essentially just the theory of $\mathbb{R}$ as an ordered group, it is straightforward to check that T is o-minimal and that T has elimination of imaginaries and a modular lattice of algebraically closed sets. Hence $\mathop{\smile\!\!\!\!\!|}^{\text{þ}}$ is a canonical independence relation for T^{eq}.

Now consider T', the theory of the following variant $\mathcal{R}'$ of $\mathcal{R}$: Instead of f we have a partial function f' which is the restriction of f to the interval $(-1, 1)$ of $\mathbb{R}_<$. Everything else is as in the definition of $\mathcal{R}$. Note that T' is essentially a reduct of T.

Let $\overline{\mathcal{R}}$ be a big elementary extension of $\mathcal{R}$ and $\overline{\mathcal{R}}'$ its 'reduct' to the signature of T'. Let $\overline{\mathcal{R}}_< = \{a \in \overline{\mathcal{R}} \mid a < *\} \supseteq \mathbb{R}_<$ and $\overline{\mathcal{R}}_> = \{a \in \overline{\mathcal{R}} \mid a > *\} \supseteq \mathbb{R}_>$. The (algebraic) closure of a set A in $\overline{\mathcal{R}}'$ turns out to be the smallest set $C \supseteq A$ containing the five constants and such that $C \cap \overline{\mathcal{R}}_<$ and $C \cap \overline{\mathcal{R}}_>$ are divisible subgroups of $\overline{\mathcal{R}}_<$ or $\overline{\mathcal{R}}_>$, respectively, and which is closed under f' and f'^{-1}. Since every closed subset of a model of T' is an elementary submodel, T' has elimination of imaginaries by [Pil86, Proposition 3.2].

Yet $\mathop{\smile\!\!\!\!\!|}^{\text{þ}}$ for T' does not have the intersection property: Choose an element a such that $\mathbb{R}_< < a < *$, and let $b = f(a)$. Note that $\dim(ab) = 1$ when evaluated in T, but $\dim(ab) = 2$ when evaluated in T'. Choose ϵ_1, ϵ_2 in the interval $(-1, 1)$ of $\overline{\mathcal{R}}_<$ such that $\dim(\epsilon_1\epsilon_2) = 2$, and set $a_i = a + \epsilon_i$, $b_i = f(a_i)$.

From now on all calculations will be in T'. Now $b = b_1 - f'(a_1 - a) \in \operatorname{cl}(a_1b_1a)$ where cl is (algebraic) closure in T', and it is only a matter of linear algebra to check that we have $b \notin \operatorname{cl}(a_1b_1a_2b_2)$.

Hence $\dim(ab/a_1b_1a_2b_2) = \dim(ab/a_1b_1) = \dim(ab/a_2b_2) = 1$, and therefore $ab \mathop{\smile\!\!\!\!\!|}^{\text{þ}}_{a_1b_1} a_1b_1a_2b_2$ and also $ab \mathop{\smile\!\!\!\!\!|}^{\text{þ}}_{a_2b_2} a_1b_1a_2b_2$. Yet $\operatorname{cl}(a_1b_1) \cap \operatorname{cl}(a_2b_2) = \operatorname{cl}\emptyset$, and $ab \not\mathop{\smile\!\!\!\!\!|}^{\text{þ}}_{\emptyset} a_1b_1a_2b_2$ because $b \in \operatorname{cl}(a_1b_1a)$ and $b \notin \operatorname{cl}(a)$. Therefore $ab \not\mathop{\smile\!\!\!\!\!|}^{\text{þ}}_{\operatorname{cl}(a_1b_1)\cap\operatorname{cl}(a_2,b_2)} a_1b_1a_2b_2$ witnesses that the intersection property fails for $\operatorname{tp}(ab/\operatorname{cl}(a_1b_1a_2b_2))$.

Thus T' is in fact an o-minimal theory for which $\mathop{\smile\!\!\!\!\!|}^{\text{þ}}$ is not a canonical independence relation, and T' is the reduct of another o-minimal theory for which $\mathop{\smile\!\!\!\!\!|}^{\text{þ}}$ is canonical. Therefore $\mathop{\smile\!\!\!\!\!|}^{\text{þ}}$ for T^{eq} need not be canonical for rosy theories, in fact not even for o-minimal theories; and having a canonical independence relation is not preserved under reducts.

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